

# Chapter 3

## The Scaling Hypothesis

Previously, we found that singular behaviour in the vicinity of a second order critical point was characterised by a set of critical exponents  $\{\alpha, \beta, \gamma, \delta, \dots\}$ . These power law dependencies of thermodynamic quantities are a symptom of **scaling behaviour**. Mean-field estimates of the critical exponents were found to be unreliable due to fluctuations. However, since the various thermodynamic quantities are related, these exponents can not be completely independent of each other. The aim of this chapter is to employ scaling ideas to uncover relationships between them.

### 3.1 Homogeneity

The non-analytic structure of the Ginzburg-Landau model was found to be a coexistence line for  $t < 0$  and  $h = 0$  that terminates at the critical point  $h = t = 0$ . Thermodynamic quantities  $Q(t, h)$  in the vicinity of the critical point are characterised by various exponents. In particular, within the saddle-point approximation we found that the free energy density was given by

$$f \equiv \frac{\beta F}{V} = \min_{\mathbf{m}} \left[ \frac{t}{2} \mathbf{m}^2 + u \mathbf{m}^4 - \mathbf{h} \cdot \mathbf{m} \right] \sim \begin{cases} -t^2/u, & h = 0, t < 0, \\ -h^{4/3}/u^{1/2} & h \neq 0, t = 0. \end{cases} \quad (3.1)$$

In fact, the free energy can be described by a single **homogeneous** function<sup>1</sup> in  $t$  and  $h$

$$f(t, h) = t^2 g_f(h/t^\Delta) \quad (3.2)$$

where  $\Delta$  is known as the ‘‘Gap exponent’’. Comparison with Eq. (3.1) shows that, if we set  $\Delta = 3/2$ , the correct asymptotic behaviour of  $f$  is obtained,

$$\begin{aligned} \lim_{x \rightarrow 0} g_f(x) &\sim -\frac{1}{u}, & f(t, h = 0) &\sim -\frac{t^2}{u}, \\ \lim_{x \rightarrow \infty} g_f(x) &\sim \frac{x^{4/3}}{u^{1/3}}, & f(t = 0, h) &\sim t^2 \left( \frac{h}{t^\Delta} \right)^{4/3} \sim h^{4/3}. \end{aligned}$$

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<sup>1</sup>A function  $f(x)$  is said to be homogeneous of degree  $k$  if it satisfies the relation  $f(x) = b^k f(bx)$ .

The assumption of homogeneity is that, on going beyond the saddle-point approximation, the singular form of the free energy (and of any other thermodynamic quantity) retains a homogeneous form

$$\boxed{f_{\text{sing.}}(t, h) = t^{2-\alpha} g_f \left( \frac{h}{t^\Delta} \right)} \quad (3.3)$$

where the actual exponents  $\alpha$  and  $\Delta$  depend on the critical point being considered.

**Heat Capacity:** For example, the dependence on  $t$  is chosen to reproduce the heat capacity singularity at  $h = 0$ . The singular part of the energy is obtained from

$$E_{\text{sing.}} \sim \frac{\partial f}{\partial t} \sim (2 - \alpha)t^{1-\alpha} g_f(h/t^\Delta) - \Delta h t^{1-\alpha-\Delta} g'_f(h/t^\Delta) \equiv t^{1-\alpha} g_E(h/t^\Delta),$$

where the prime denotes the derivative of the function with respect to the argument. Thus, the derivative of one homogeneous function is another. Similarly, the second derivative takes the form

$$C_{\text{sing.}} \sim -\frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha} g_C(h/t^\Delta),$$

reproducing the scaling  $C_{\text{sing.}} \sim t^{-\alpha}$  as  $h \rightarrow 0$ .<sup>2</sup>

**Magnetisation:** Similarly the magnetisation is obtained from Eq. (3.3) using the expression

$$m(t, h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^\Delta).$$

In the limit  $x \rightarrow 0$ ,  $g_m(x)$  is a constant, and  $m(t, h = 0) \sim t^{2-\alpha-\Delta}$  (i.e.  $\beta = 2 - \alpha - \Delta$ ). On the other hand, if  $x \rightarrow \infty$ ,  $g_m(x) \sim x^p$ , and  $m(t = 0, h) \sim t^{2-\alpha-\Delta}(h/t^\Delta)^p$ . Since this limit is independent of  $t$ , we must have  $p\Delta = 2 - \alpha - \Delta$ . Hence  $m(t = 0, h) \sim h^{(2-\alpha-\Delta)/\Delta}$  (i.e.  $\delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta$ ).

**Susceptibility:** Finally, calculating the susceptibility we obtain

$$\chi(t, h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^\Delta) \Rightarrow \chi(t, h = 0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha.$$

Thus the consequences of homogeneity are:

- The singular parts of all critical quantities,  $Q(t, h)$  are homogeneous, with the same exponents above and below the transition.
- Because of the interconnections via thermodynamic derivatives, the same gap exponent,  $\Delta$  occurs for all such quantities.
- All critical exponents can be obtained from only *two* independent ones, e.g.  $\alpha$ ,  $\Delta$ .

<sup>2</sup>It may appear that we have the freedom to postulate a more general form,  $C_\pm = t^{-\alpha_\pm} g_\pm(h/t^\Delta_\pm)$  with different functions for  $t > 0$  and  $t < 0$  that match at  $t = 0$ . However, this can be ruled out by the condition that the free energy is analytic everywhere except on the coexistence line  $h = 0$  and  $t < 0$ .

- As a result of above, one obtains a number of *exponent identities*:

$$\boxed{\alpha + 2\beta + \gamma = 2} \quad (\text{Rushbrooke's Identity})$$

$$\boxed{\delta - 1 = \gamma/\beta} \quad (\text{Widom's Identity})$$

These identities can be checked against the following table of critical exponents. The first three rows are based on a number of theoretical estimates in  $d = 3$ ; the last row comes from an exact solution in  $d = 2$ . The exponent identities are approximately consistent with these values, as well as with all reliable experimental data.

			$\alpha$	$\beta$	$\gamma$	$\delta$	$\nu$	$\eta$
$d = 3$	$n = 1$	Ising	0.12	0.31	1.25	5	0.64	0.05
	$n = 2$	XY-spin	0.00	0.33	1.33	5	0.66	0.00
	$n = 3$	Heisenberg	-0.14	0.35	1.4	5	0.7	0.04
$d = 2$	$n = 1$	Ising	0	1/8	7/4	15	1	1/4

## 3.2 Hyperscaling and the Correlation Length

The homogeneity assumption relates to the free energy and quantities derived from it. It says nothing about correlation functions. An important property of a critical point is the divergence of the correlation length, which is responsible for (and can be deduced from) the divergence of response functions. In order to obtain an identity involving the exponent  $\nu$  describing the divergence of the correlation length, we replace the homogeneity assumption for the free energy with the following *two* conditions:

1. The correlation length has a homogeneous form,

$$\boxed{\xi(t, h) \sim t^{-\nu} g_{\xi} \left( \frac{h}{t^{\Delta}} \right)}$$

For  $t = 0$ ,  $\xi$  diverges as  $h^{-\nu_h}$  with  $\nu_h = \nu/\Delta$ .

2. Close to criticality, the correlation length  $\xi$  is the most important length scale, and is *solely* responsible for singular contributions to thermodynamic quantities.

The second condition determines the singular part of the free energy. Since  $\ln \mathcal{Z}(t, h)$  is *dimensionless* and *extensive* (i.e. scales in proportion with the volume  $L^d$ ), it must take the form

$$\ln \mathcal{Z} = \left( \frac{L}{\xi} \right)^d \times g_s + \left( \frac{L}{a} \right)^d \times g_a,$$

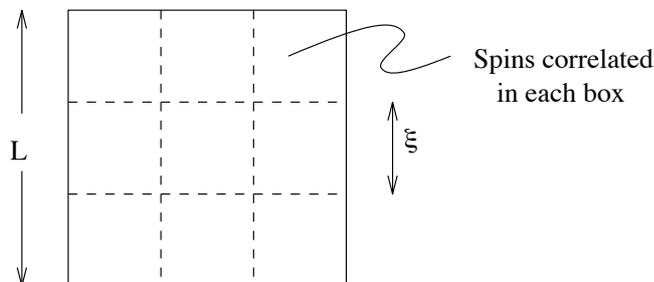


Figure 3.1: Within each cell of size  $\xi$  spins are correlated.

where  $g_s$  and  $g_a$  are non-singular functions of dimensionless parameters ( $a$  is an appropriate microscopic length). The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t, h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^\Delta). \quad (3.4)$$

A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length (Fig. 3.2). Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as  $(L/\xi)^d$ .

The consequences of Eq. (3.4) are:

- Homogeneity of  $f_{\text{sing.}}$  emerges naturally.
- We obtain the additional exponent relation

$$\boxed{2 - \alpha = d\nu} \quad (\text{Josephson's Identity})$$

Identities obtained from the generalised homogeneity assumption involve the space dimension  $d$ , and are known as **hyperscaling relations**. The relation between  $\alpha$  and  $\nu$  is consistent with the exponents in the table above. However, it does not agree with the mean-field values,  $\alpha = 0$  and  $\nu = 1/2$ , which are valid for  $d > 4$ . Any theory of critical behaviour must therefore account for the validity of this relation in low dimensions, and its breakdown in  $d > 4$ .

### 3.3 Correlation Functions and Self-Similarity

So far we have not accounted for the exponent  $\eta$  which describes the decay of correlation functions at criticality. Exactly at the critical point the correlation length is infinite, and there is no other length scale to cut-off the decay of correlation functions. Thus all correlations decay as a power of the separation. As discussed in the previous chapter, the magnetisation falls off as

$$\boxed{G_c(\mathbf{x}) \equiv \langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0) \rangle - \langle \mathbf{m}^2 \rangle \sim \frac{1}{|\mathbf{x}|^{d-2+\eta}}}$$

where  $\eta$  was deduced from the form factor.

Away from criticality, the power laws are cut-off for distances  $|\mathbf{x}| \gg \xi$ . As the response functions can be obtained from integrating the connected correlation functions, there are additional exponent identities such as Fisher's identity

$$\chi \sim \int d^d \mathbf{x} G_c(\mathbf{x}) \sim \int_{|\mathbf{x}| \leq \xi} \frac{d^d \mathbf{x}}{|\mathbf{x}|^{d-2+\eta}} \sim \xi^{2-\eta} \sim t^{-\nu(2-\eta)} \implies \boxed{\gamma = (2 - \eta)\nu}$$

Therefore, two *independent* exponents are sufficient to describe all singular critical behaviour.

An important consequence of these scaling ideas is that the critical system has an additional **dilation symmetry**. Under a change of scale, the critical correlation functions behave as

$$\boxed{G_{\text{critical}}(\lambda \mathbf{x}) = \lambda^p G_{\text{critical}}(\mathbf{x})}$$

This implies a **scale invariance** or **self-similarity**: If a snapshot of the critical system is enlarged by a factor of  $\lambda$ , apart from a change of contrast ( $\lambda^p$ ), the resulting snapshot is statistically similar to the original. Such statistical self-similarity is the hallmark of **fractal** geometry. The Ginzburg-Landau functional was constructed on the basis of **local** symmetries such as rotational invariance. If we could add to the list of constraints the requirement of dilation symmetry, the resulting probability would indeed describe the critical point. Unfortunately, it is not in general possible to see directly how such a requirement constrains the effective Hamiltonian.<sup>3</sup> We shall instead prescribe a less direct route by following the effects of the dilation operation on the effective energy; a procedure known as the **renormalisation group**.

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<sup>3</sup>One notable exception is in  $d = 2$ , where dilation symmetry implies conformal symmetry.

