Continuous symmetry breaking and Goldstone modes

Ginzburg-Landau Hamiltonian for $n \geq 2$

$$\beta H[\underline{m}] = \int d^d \underline{x} \ \frac{t}{2} m^2 + u m^4 + \frac{K}{2} |\nabla \underline{m}|^2 - \underline{h} \cdot \underline{m}$$

i). For $\underline{h} = \underline{0}$, βH has full global rotational symmetry in spin space.

— but ground state is ferromagnetic

I.e. At $T < T_c$, appearance of long range order (LRO) is accompanied by spontaneous symmetry breaking

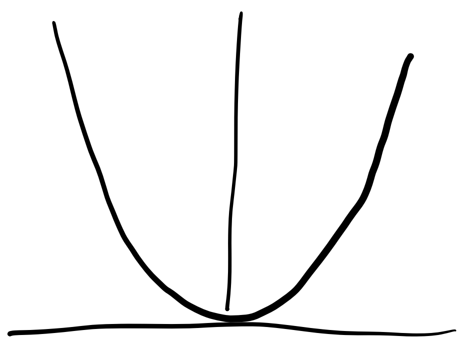
ii) Uniform (i.e. global) spin rotations cost no energy - but requires motion of whole system.

iii) A slowly varying (in space) rotation costs very little energy

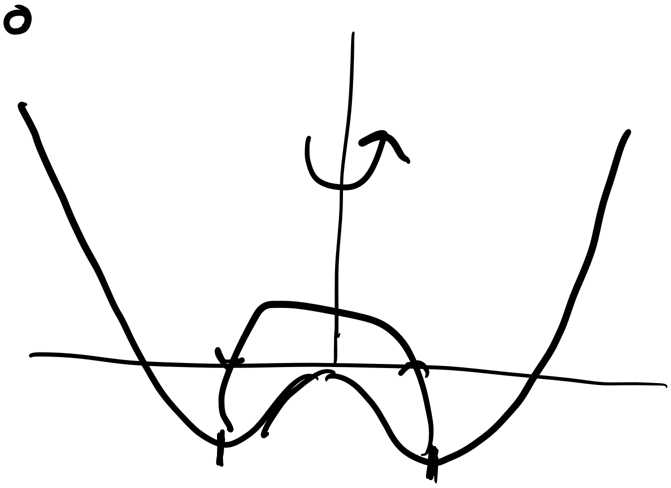
- form low-energy collective excitations

e.g. magnons (spin waves)
phonons

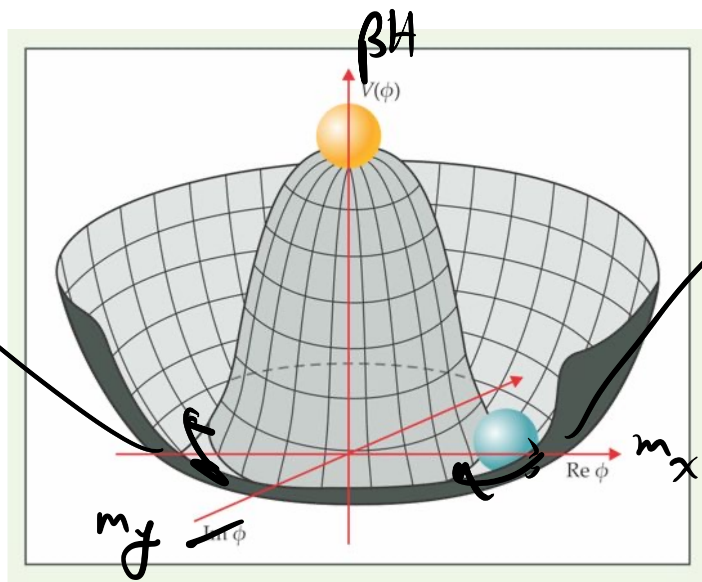
$t>0$



$t<0$



transverse mode
phase mode



longitudinal amplitude modes are massive cost energy

Generally, the breaking of a continuous symmetry

$\Rightarrow$ existence of low energy excitations

— Goldstone modes

Goldstone modes have a dramatic influence on the nature of LRO in low dimensions:

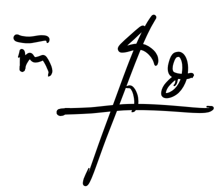
To see this, consider $n=2$ (XY-spins, superfluids)

Z_2 − Ising $n=1$

$$\uparrow\ \uparrow\ \uparrow\ \vdots\ \downarrow\ \downarrow\ \downarrow$$

domain walls

$T<T_c$



$$\underline{m} = \bar{m}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\underline{\nabla}\,\underline{m} = \bar{m}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}\underline{\nabla}\theta$$

$$\beta H[\theta] = \beta H_0 + \frac{\bar{K}}{2}\int d^d\underline{x}\ |\underline{\nabla}\theta|^2$$

$$\bar{K} = K\bar{m}^2$$

$$Z = \int \mathcal{D}\theta(\underline{x})\ e^{-\beta H[\theta]}$$

- looks Gaussian but note $\theta(\underline{x})$ is periodic 2π.
  - this means there are constraints on our integral and this admits topologically non-trivial field configurations (vortices).

Approximation for large $\bar{K}$ (i.e. low T) we assume that the contribution to Z is small (true $d>2$!)

θ is unconstrained!

$$\langle\theta\rangle = 0$$

$$\langle\theta(\underline{x})\theta(\underline{x}')\rangle = G(\underline{x}-\underline{x}') = -\frac{C_d(x-x')}{\bar{K}}$$

where $\nabla^2 c_d(x) = \delta^d(x)$ — Coulomb problem in d dimension.

Applying Gauss' law

$$\int d^d x \, \nabla^2 c_d(x) = \oint d^{d-1} x \cdot \underline{\nabla} c_d$$

$$1 = S_d \, x^{d-1} \frac{d c_d}{dx}$$

surface area of a unit d-dimensional ball.

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

$$\frac{d c_d}{dx} = \frac{1}{S_d} x^{1-d}$$

$$c_d = \frac{x^{2-d}}{(2-d) S_d} + \text{Const.}$$

So $\langle [\theta(x) - \theta(0)]^2 \rangle = 2\left[\langle \theta^2(0) \rangle - \langle \theta(\underline{x}) \theta(\underline{0}) \rangle \right]$

$$= \frac{2\,(x^{2-d} - a^{2-d})}{\bar{K}\,(2-d)\,S_d}$$

lower length scale cut-off

$x \to \infty$

$$\langle [\theta(x) - \theta(0)]^2 \rangle \to \begin{cases} \text{const} & d > 2 \quad x \to 0 \\ \text{log div} & d = 2 \\ \text{diverges} & d < 2 \end{cases}$$

[N.B. For dimensions $d \geq 2$, there appears a short length scale or UV divergence. In principle, this could be controlled by modifying the Hamiltonian at short-length scales (lattice regularisation).

However, since we know that there is an implicit lattice scale, we know the divergence is unphysical – so we can neglect it.]

<u>Consequences</u>

$$\langle \underline{m}(\underline{x}) \cdot m(\underline{0}) \rangle = \bar{m}^2 \, \mathrm{Re} \langle e^{i[\theta(\underline{x}) - \theta(\underline{0})]} \rangle$$

$$= \bar{m}^2 \exp\left[-\frac{1}{2}\langle[\theta(x)-\theta(0)]^2\rangle\right]$$

$$\xrightarrow[x\to\infty]{} \begin{cases} \bar{m}^2 & d>2 \\ 0 & d\leq 2 \end{cases}$$

True LRO should approach a finite constant

If zero, fluctuations destroyed LRO.

<u>Mermin–Wagner theorem</u>

For system with continuous symmetry (and short ranged interaction), there is <u>no</u> LRO in dimension $d\leq 2$ for any finite temperature.

Thermal fluctuations destroy LRO.

$d=2$ is known as lower critical dimension.

(for discrete symmetry, lower critical dimension $d=1$.)

Non-examinable

Non-examinable derivation of S_d

Volume $\quad V \propto r^d$

Surface area $\quad S = S_d\, r^{d-1}$

$$\int_{-\infty}^{\infty} \prod_i^d dx_i\, e^{-x_i^2} = \pi^{\frac{d}{2}} = \int_0^{\infty} dr\; S_d\, r^{d-1} e^{-r^2}$$

$$\Gamma(n) = \int_0^{\infty} dt\; e^{-t}\, t^{n-1}$$

$$\pi^{\frac{d}{2}} = S_d \cdot \frac{1}{2}\Gamma\left(\frac{d}{2}\right)$$

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$

$$\int_0^{\infty} dr\; r^{d-1} e^{-r^2} = \frac{1}{2}\int_0^{\infty} dt\; t^{\frac{d}{2}-1} e^{-t} \qquad t = r^2$$

$$dt = 2r\, dr$$

$$= \frac{1}{2}\Gamma\left(\frac{d}{2}\right) \qquad r^{d-2} = t^{\frac{d}{2}-1}$$