Quantum Phase Transitions

Dualities

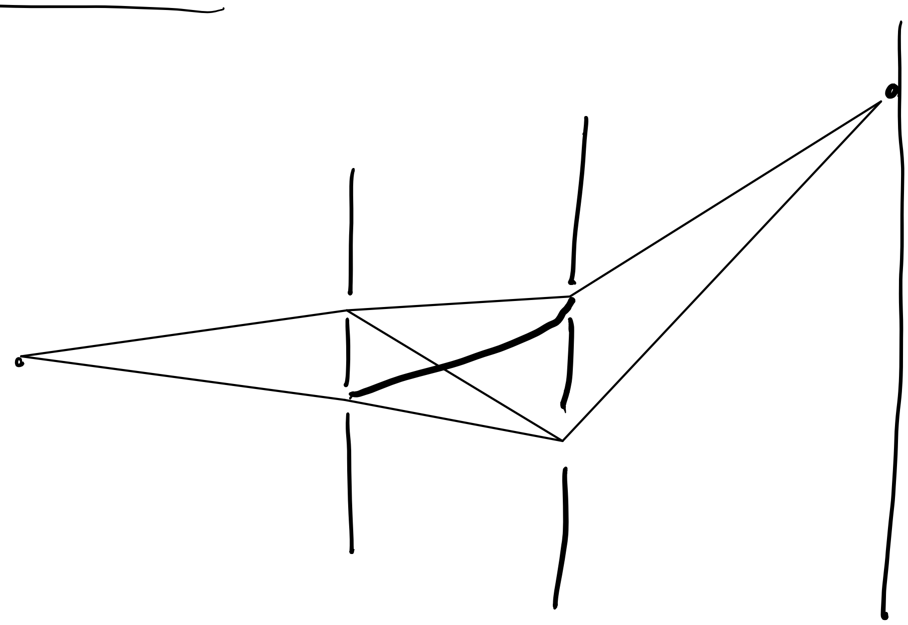
$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_0} \qquad \underline{\nabla}\cdot\underline{B} = 0$$

$$\underline{\nabla}\times\underline{E} = -\dot{\underline{B}} \qquad \underline{\nabla}\times\underline{B} = \mu_0\underline{J} + \frac{1}{c^2}\dot{\underline{E}}$$

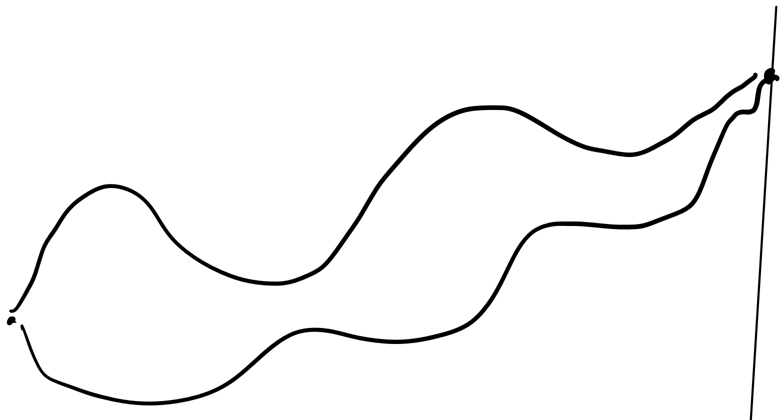
AdS/CFT – holographic

Quantum - Classical mapping

Intuitive Picture



$e^{iS/\hbar}$



"sum over histories"

or "path integral"

$$[\hat{x}, \hat{p}] = i\hbar \leftarrow \hbar = 1$$

$$x\,\delta'(x) = -\delta(x)$$

$$\langle x | [\hat{x}, \hat{p}] | x' \rangle = i\hbar \langle x | x' \rangle = i\hbar\, \delta(x - x')$$

$$\langle x | \hat{x}\hat{p} - \hat{p}\hat{x} | x' \rangle = (x - x') \langle x | \hat{p} | x' \rangle$$

$$\Rightarrow \quad \langle x | \hat{p} | x' \rangle = -i\hbar \frac{\partial}{\partial x} \delta(x - x')$$

$$\psi_p(x) = \langle x | p \rangle$$

$$\hat{p}\,|p\rangle = p\,|p\rangle$$

$$\Rightarrow \quad \langle x|\hat{p}|p\rangle = p\langle x|p\rangle$$

$$I = \int dx' \, |x'\rangle\langle x'|$$

$$\int dx' \, \langle x|\hat{p}|x'\rangle\langle x'|p\rangle = p\langle x|p\rangle$$

$$\Rightarrow \quad \int dx' \, -i\hbar\frac{\partial}{\partial x}\delta(x-x')\langle x'|p\rangle = p\langle x|p\rangle$$

$$\Rightarrow \quad -i\hbar\frac{\partial}{\partial x}\int dx' \, \delta(x-x')\langle x'|p\rangle = p\langle x|p\rangle$$

$$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\langle x|p\rangle$$

$$\psi_p(x) = \langle x|p\rangle = \frac{e^{\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}}$$

<u>Path integral for quantum partition function</u>

$$\hat{H} = \sum_{i=1}^{N}\frac{p_i^2}{2m} + \hat{V}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N)$$

many body potential.

$$[\hat{x}_\nu, \hat{p}_\mu] = i\delta_{\mu\nu} \qquad d\text{-dimension}$$

$$Z = \mathrm{tr}\, e^{-\beta \hat{H}} = \int \underbrace{\prod_{i=1}^{N} d^d \underline{x}_i}_{dX} \langle \underline{x}_1, \underline{x}_2, \dots, \underline{x}_N | e^{-\beta \hat{H}} | \underbrace{\underline{x}_1, \dots \underline{x}_N}_{X} \rangle$$

$$= \int dX \langle X | e^{-\beta \hat{H}} | X \rangle \qquad X^2 = \sum x_i^2$$

$$X \cdot P = \sum \underline{x}_i \cdot p_i$$

$$= \int dX \langle X | e^{-\frac{\beta}{M}\hat{H}} \, e^{-\frac{\beta}{M}\hat{H}} \cdots e^{-\frac{\beta}{M}\hat{H}} | X \rangle$$

$\int dX_1 |X_1\rangle\langle X_1|$ "2" "3" "M"

$$\varepsilon = \frac{\beta}{M}$$

$$\langle X_1 | e^{-\varepsilon \hat{H}} | X_2 \rangle \langle X_2 | e^{-\varepsilon \hat{H}} | X_3 \rangle \cdots \qquad \langle X_M | e^{-\varepsilon \hat{H}} | X_1 \rangle$$

$\int dP_1 |P_1\rangle\langle P_1|$ "2" "M"

Key bracket:

$$\langle P_1 | e^{-\varepsilon \hat{H}} | X_{i+1} \rangle$$

$$= \langle P_1 | (1 - \varepsilon \hat{H} | X_{i+1} \rangle$$

$$= \left(1 - \varepsilon\left(\frac{p_i^2}{2m} + V(x_{i+1})\right)\right) \langle p_i | x_{i+1} \rangle$$

$$= e^{-\varepsilon\left(\frac{p_i^2}{2m} + V(x_{i+1})\right)} \; e^{-i p_i \cdot x_{i+1}}$$

So putting back together

$$Z = \int \prod_{i=1}^{N} dx_i \int \prod_{i=1}^{N} dp_i \, \exp\left\{ -i \sum p_i \cdot (x_{i+1} - x_i) - \varepsilon \sum \frac{p_i^2}{2m} + V(x_i) \right\}$$

Note the periodic b.c.

$$x_{N+1} = x_1$$

Taking the continuum limit

$$x_i \to x(\tau) \qquad \tau = (i-1)\varepsilon$$

and we find the functional integral

$$Z = \int_{x(\beta) = x(0)} \mathcal{D}x(\tau) \int \mathcal{D}p(\tau) \, e^{-\int_0^\beta d\tau \left(i\, p(\tau)\, \partial_\tau x + \frac{p^2}{2m} + V(x(\tau)) \right)}$$

Integrating out momentum fields

$$\mathcal{Z} = \int\limits_{\underline{x}_i(\beta) = \underline{x}_i(0)} \mathcal{D}\underline{x}_i(\tau)\, e^{-H[\underline{x}_i(\tau)]}$$

$$H[\underline{x}_i(\tau)] = \int_0^\beta d\tau \sum_{i=1}^{N} \frac{m|\partial_\tau x_i|^2}{2} + V[x_i(\tau)]$$

$\uparrow$

classical Hamiltonian.

Quantum - Classical Mapping

A d-dimensional quantum system can be mapped onto a d+1 dimensional classical system. Extra dimension is the imaginary time τ.

Key correspondences of the Quantum-Classical mapping

Correspondence	Quantum	Classical
Hamiltonian $\hat{H} \leftrightarrow \beta H$	$\hat{H} = \sum_i \frac{\hat{p}_i^2}{2M} + V[\hat{x}_i]$, where $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$	$\beta H = \int_0^\beta \mathrm{d}\tau \left[\sum_i \frac{M}{2}\left(\frac{\partial m_i(\tau)}{\partial \tau}\right)^2 + V[m_i(\tau)]\right]$
Order Parameter $\hat{x}_i \leftrightarrow m_i(\tau)$	$\hat{x}_i$	$m_i(\tau)$
Partition function $\mathcal{Z}_\mathrm{Q} = \mathcal{Z}_\mathrm{C}$	$\mathcal{Z}_\mathrm{Q} = \sum_n \langle n\vert e^{-\beta\hat{H}}\vert n\rangle$	$\mathcal{Z}_\mathrm{C} = \int_{m_i(0)=m_i(\beta)} \mathcal{D}m_i(\tau)\, e^{-\beta H[m_i(\tau)]}$
Expectation Value $\langle \hat{x}_i\rangle = \langle m_i(0)\rangle$	$\langle \hat{x}_i\rangle = \frac{1}{\mathcal{Z}_\mathrm{Q}} \sum_n \langle n\vert e^{-\beta\hat{H}}\hat{x}_i\vert n\rangle$	$\langle m_i(0)\rangle = \frac{1}{\mathcal{Z}_\mathrm{C}} \int_{m_i(0)=m_i(\beta)} \mathcal{D}m_i(\tau)\, m_i(0) e^{-\beta H[m_i(\tau)]}$
Correlator $\langle e^{\hat{H}\tau}\hat{x}_j e^{-\hat{H}\tau}\hat{x}_i\rangle = \langle m_j(\tau)m_i(0)\rangle$, where $\tau > 0$	$\langle e^{\hat{H}\tau}\hat{x}_j e^{-\hat{H}\tau}\hat{x}_i\rangle = \frac{1}{\mathcal{Z}_\mathrm{Q}}\sum_n \langle n\vert e^{-\beta\hat{H}} e^{\hat{H}\tau}\hat{x}_j e^{-\hat{H}\tau}\hat{x}_i\vert n\rangle$	$\langle m_j(\tau)m_i(0)\rangle = \frac{1}{\mathcal{Z}_\mathrm{C}} \int_{m_i(0)=m_i(\beta)} \mathcal{D}m_i(\tau)\, m_j(\tau)m_i(0)e^{-\beta H[m_i(\tau)]}$

Notes:

Example

S. H. O. $\longleftrightarrow$ 1d Ising model

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2} M\omega^2 \hat{x}^2 \longleftrightarrow \beta H[m(\tau)]$$

$$= \int_0^\beta d\tau \ \frac{1}{2} M \left(\frac{\partial m}{\partial \tau}\right)^2 + \frac{1}{2} M \omega^2 m^2$$

By this correspondence

$$K = M \quad \text{and} \quad t = M\omega^2$$

Recall $\qquad K \xi^{-2} = t$

Correlation length $\quad \xi = \sqrt{\frac{K}{t}} = \sqrt{\frac{M}{M\omega^2}}$

$$= \frac{1}{\omega} = \frac{\hbar}{\Delta E}$$

General result: Correlation length (along imaginary time) is the inverse of the energy gap between the ground state and 1st excited state.