Wilson's Perturbative RG

$$\beta H = \beta H_0 + U \quad \text{perturbation}$$

Gaussian

1. Coarse grain

$$\underline{m}(\underline{q}) = \begin{cases} \underline{m}_<(\underline{q}) & 0 < q < \frac{\Lambda}{b} \\ \underline{m}_>(\underline{q}) & \frac{\Lambda}{b} < q < \Lambda \end{cases}$$

Modes decouple at Gaussian order βH_0

$$Z = \int \mathcal{D}m_< \mathcal{D}m_> \exp\left[-\beta H_0[m_<] - \beta H_0[m_>] - U[m_>, m_<]\right]$$

$$= \int \mathcal{D}m_< \, e^{-\beta H_0[m_<]} \frac{\int \mathcal{D}m_> \, e^{-\beta H_0[m_>]} e^{-U}}{Z_{0,>}} Z_{0,>}$$

$Z_{0,>}$: $\beta H_0[m_>]$

$$\frac{\int \mathcal{D}m_> \, e^{-\beta H_0[m_>]} e^{-U}}{Z_{0,>}} = \left\langle e^{-U[m_>, m_<]} \right\rangle_{m_>}$$

$$= \int \mathcal{D}m_< \, e^{-\left[\beta H_0[m_<] - \log Z_{0,>} - \log \langle e^{-U}\rangle_{m_>}\right]}$$

$$\downarrow$$

$$\beta H[m_<]$$

renormalized by fast fluctuations.

Cumulant expansion

$$\log \langle e^{-u} \rangle = -\langle u \rangle + \frac{1}{2}\left(\langle u^2 \rangle - \langle u \rangle^2 \right) - \cdots$$

negative shift to βH_0.

Recall $$\beta H_0[m_>] = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \, \frac{1}{2}(t + Kq^2) \, |m_>(q)|^2$$

$$\frac{d^d q}{(2\pi)^d} \to (dq)$$

$$\langle m^>_\alpha(q) \, m^>_\beta(q') \rangle_{m_>} = \delta_{\alpha\beta} (2\pi)^d \, \delta(q + q') \, G_0(q)$$

$$G_0(q) = \frac{1}{t + Kq^2}.$$

First order $\langle U \rangle_{m_>}$

$$U = u \int d^d \underline{x} \; m(x)^4 \qquad (\underline{m}\cdot\underline{m})^2$$

$$= u \int (dq_1)(dq_2)(dq_3)(dq_4) \, d^d \underline{x} \; e^{i(q_1+q_2+q_3+q_4)\cdot \underline{x}} \; m^4$$

$$= u \int (dq_1)(dq_2)(dq_3)(dq_4)\,(2\pi)^d\,\delta(q_1+q_2+q_3+q_4)$$

$$\times \left(m_>(q_1) + m_<(q_1)\right)(\text{"2"})(\text{"3"})(\text{"4"})$$

Only terms even in $m_>$ contribute to average over $m_>$.

Odd $\langle m_>^{\text{odd}} \rangle = 0$ by symmetry.

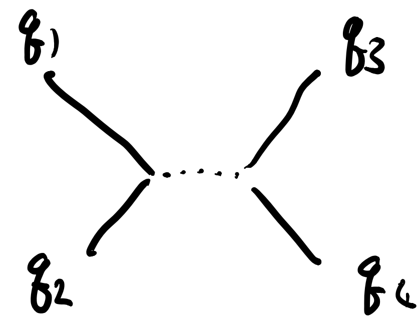
<u>Feynman Diagrams</u>

$m_>$

$m_<$

$$c_1 = \langle m_<(q_1)\cdot m_<(q_2)\; m_<(q_3)\cdot m_<(q_4)\rangle_{m_>}$$

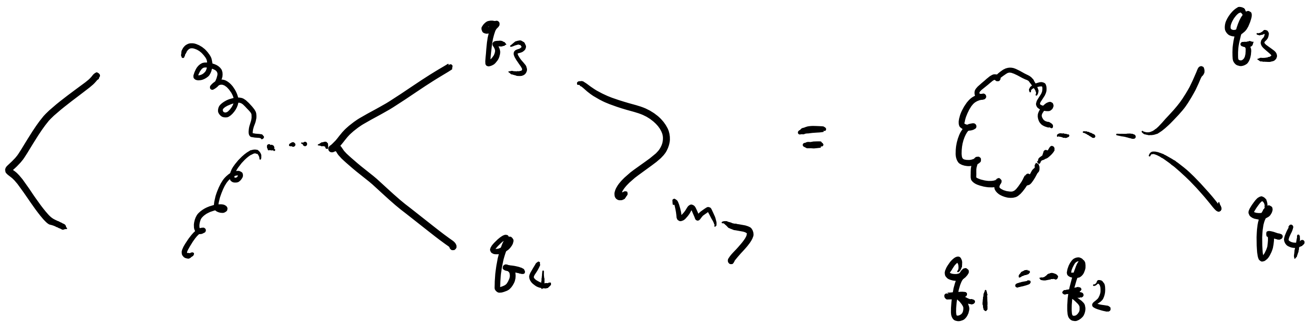
$$\rightarrow u[m_<]$$



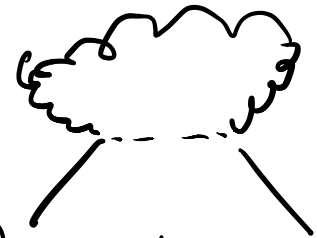
$$c_2 = \langle m_>(q_1)\cdot m_>(q_2)\; m_<(q_3)\cdot m_<(q_4)\rangle_{m_>}$$

$$= \langle m_>(q_1)\cdot m_>(q_2)\rangle_{m_>}\, m_<(q_3)\cdot m_<(q_4)$$

$$= \underset{\uparrow_n}{\delta_{\alpha\alpha}}\, G_0(q_1)\,(2\pi)^d\,\delta(q_1+q_2)\; m_<(q_3)\cdot m_<(q_4)$$



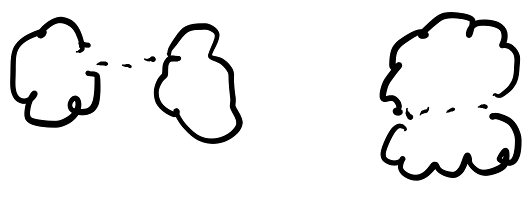
$$C_3 = \langle m_>(q_1)\cdot m_<(q_2)\; m_>(q_3)\cdot m_<(q_4)\rangle_{m_>}$$



$$= G_0(q_1)(2\pi)^d\,\delta(q_1+q_3)\; m_<(q_2)\; m_<(q_4)$$
$$\delta_{\alpha\beta}$$

$$C_4 = \langle m_>(q_1)\cdot m_>(q_2)\; m_>(q_3)\cdot m_>(q_4)\rangle_{m_>}$$

$= $ const.



Dropping "constants" (C_4 and $Z_{0,>}$), no new terms are generated

$$K \longrightarrow \tilde{K} = K$$
$$u \longrightarrow \hat{u} = u$$

$$t \rightarrow \tilde{t} = t + 2 \times 2(n+2)u \int_{\Lambda/b}^{\Lambda} (dq)\, G_0(q)$$

$2C_2$

$4C_3$

2. Rescale $\quad x' = \dfrac{x}{b} \quad ; \quad q' = b\, q$

3. Renormalise $\quad m'(q') = \dfrac{m_<(q')}{z}$

$$\beta H'[m'] = \int_0^{\Lambda} (dq')\, b^{-d} z^2 \left(\frac{\tilde{t} + \tilde{K} b^{-2} q'^2}{2} \right) |m'(q')|^2$$

$$+ \, \hat{u}\, z^4 b^{-3d} \int (dq_1)(dq_2)(dq_3)$$

$$m'(q_1') \cdot m'(q_2')\; m'(q_3') \cdot m'(-q_1 - q_2 - q_3)$$

Recursion relation

$$\begin{cases} K' = b^{-d-2} z^2 K \\ t' = b^{-d} z^2 \tilde{t} \\ u' = b^{-3d} z^4 u \end{cases}$$

Fixed point we set $K' = K$

$$\Rightarrow \quad z^2 = b^{2+d}$$

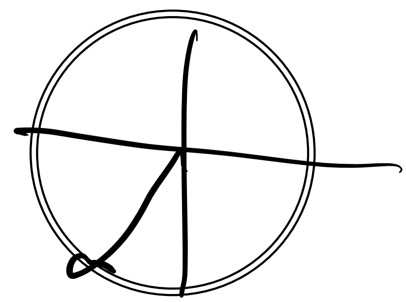
$$z = b^{1+\frac{d}{2}}$$

$$u' = b^{-3d}\, b^{4+2d}\, u = b^{4-d}\, u$$

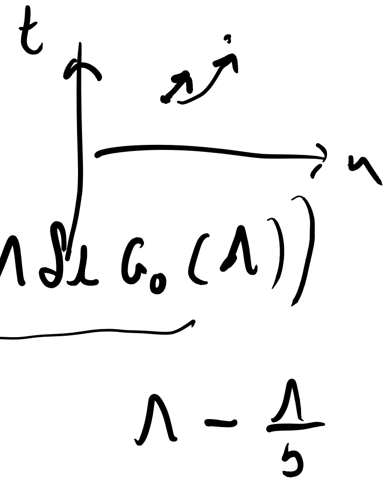
$$t' = t(b) = b^2\left[t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} (dq)\, G_0(q)\right]$$

<u>RG flow</u>

Set $b = e^{\delta l} \simeq \underline{1 + \delta l}$



$$t(b) \simeq t + \delta l \frac{dt}{dl}$$



$$= (1 + 2\delta l)\Big(t + \underbrace{4(n+2)u\, \frac{S_d}{(2\pi)^d} \Lambda^{d-1}\, \Lambda \delta l\, G_0(\Lambda)}_{u\, g(t)\, dl}\Big)$$

$$\Lambda - \frac{\Lambda}{b}$$

$$\simeq t + \big(2t + u g(t)\big) dl$$

$$\Lambda\left(1 - \frac{1}{1+\delta l}\right)$$

Similarly $\quad u(b) = (1 + (4-d)\delta l)\, u$ $\qquad \Lambda \delta l$

$$= u + \delta l \frac{du}{dl}$$

$$\frac{dt}{dl} = 2t + u\,g(t)$$

$$\frac{du}{dl} = (4-d)\,u$$

Again we have $t^* = u^* = 0$ is a fixed point.

Linearising near fixed point

$$\begin{cases} t = t^* + \delta t \\ u = u^* + \delta u \end{cases}$$

$$\frac{d}{dl}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & g(0) \\ 0 & 4-d \end{pmatrix}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix}$$

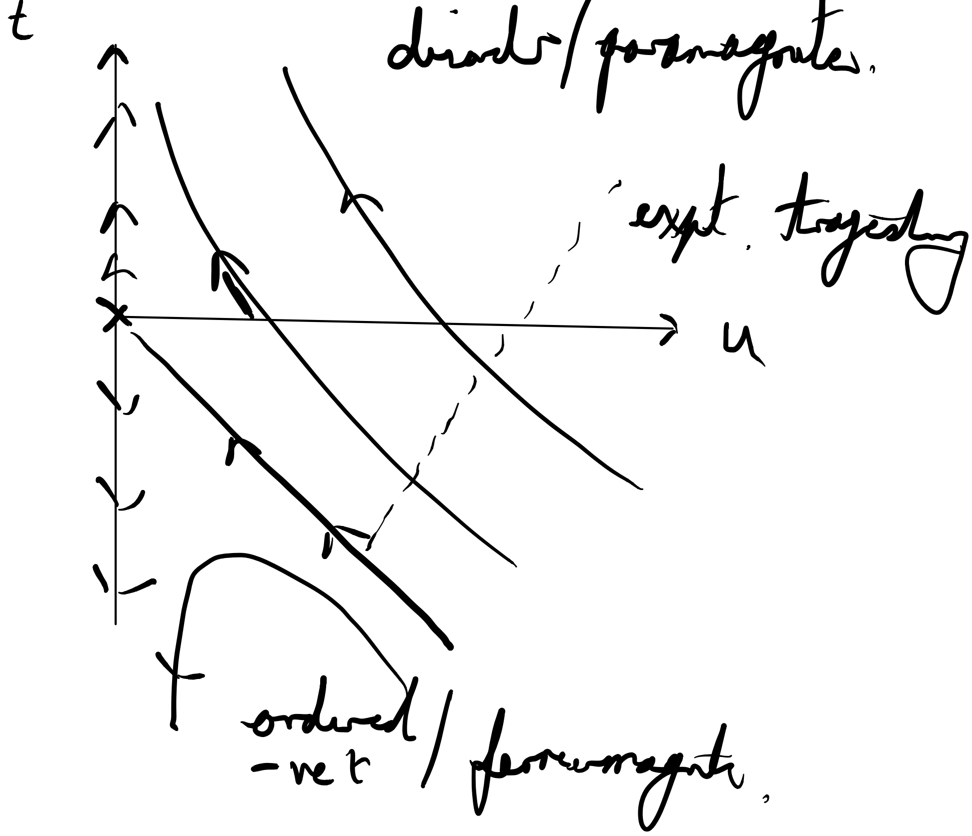
stability matrix

Eigenvalues : 2 , $4-d$. as in Gaussian theory.

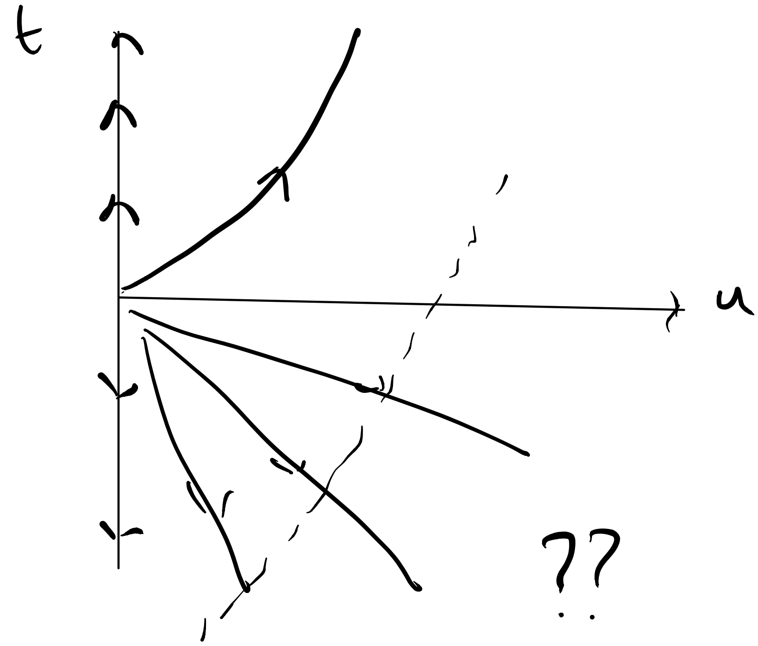
But eigenvectors 2: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and. $4-d$: $\begin{pmatrix} \frac{g(0)}{2-d} \\ 1 \end{pmatrix}$

rotated cf. Gaussian theory.

$d > 4$



$d < 4$



- Fluctuations make some of the region with $t < 0$ disordered
- $d < 4$? No new fixed point at 1st order.

Second order (non-examinable) — ε expansion

$\uparrow$ $\varepsilon = 4-d$

Since $\frac{1}{2}\left(\langle u^2\rangle - \langle u\rangle^2\right) > 0$

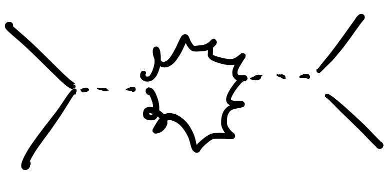
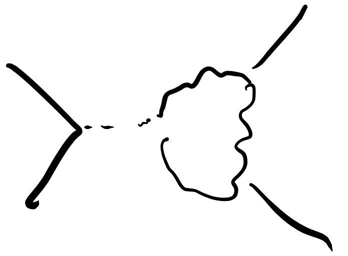
$$\frac{dt}{dl} = \left.\frac{dt}{dl}\right|_{O(u)} - Au^2$$

$$\frac{du}{dl} = \left.\frac{du}{dl}\right|_{O(u)} - Bu^2 \quad , \text{ where } A, B > 0$$

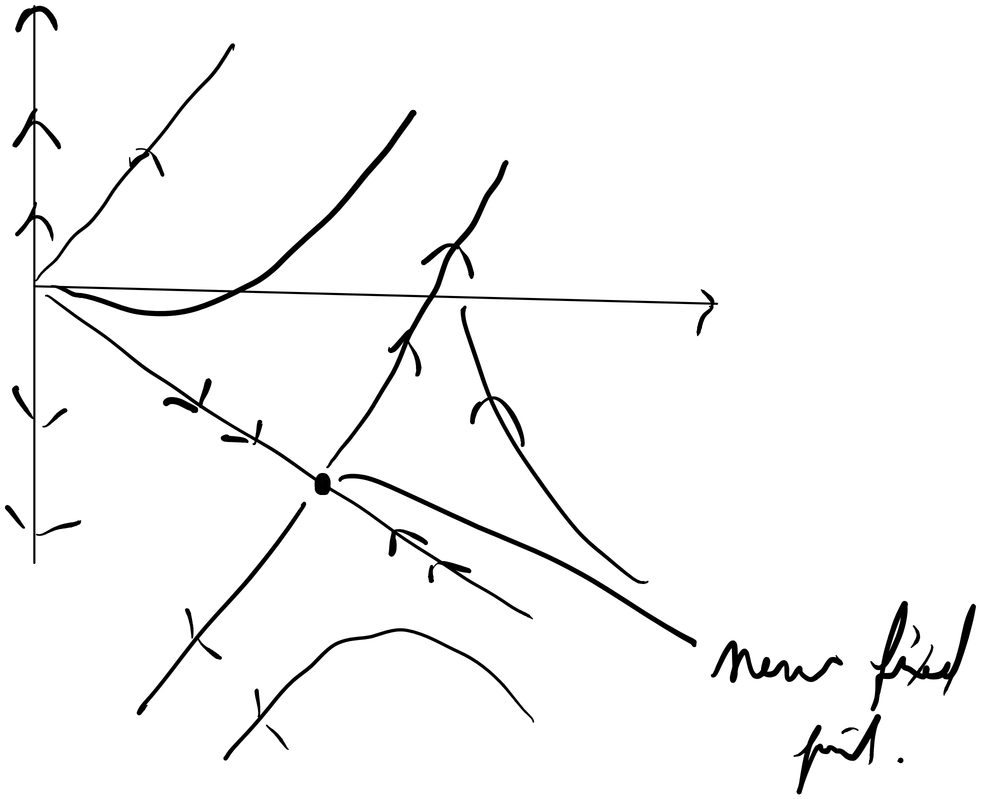
$$u^* = \frac{4-d}{B}.$$

$$t^* = -\frac{2\varepsilon}{B} g(0) + O(\varepsilon^2).$$

$$\Big[\quad B = -4(n+8)\, G_0(\Lambda)\, \Lambda^d \frac{S_d}{(2\pi)^d}$$

 ,  $\Big].$

$d < 4$.



$$y_t = 2 - \frac{n+2}{n+8}\,\varepsilon + O(\varepsilon^2)$$

$$y_u = -\varepsilon + O(\varepsilon^2)$$

$$y_h = 1 + \frac{d}{2} + O(\varepsilon^2).$$

Critical exponents are pure numbers (d, n and range of interaction)

The phase transition is controlled by the new fixed point.

K. Wilson RMP <u>55</u> 583 (1983)

Nobel lecture.