Problem Set II

Lent 2023

Perturbative Renormalisation Group Questions

1. *Perturbative RG (adpated from 2018 Part III NatSci Tripos):* Consider the long-wavelength expansion of the Hamiltonian of the 2-dimensional XY model:

$$\beta\mathcal{H}[\phi(\mathbf{r})] = \int d^2\mathbf{r} \left(\frac{K}{2} (\nabla\phi)^2 + u (\nabla\phi)^4 \right),$$

where $\phi(\mathbf{r})$ is the azimuthal angle that describes the transverse fluctuations of the magnetisation. Longitudinal fluctuations can be assumed to be frozen out. $\mathbf{r}$ spans 2-dimensional Euclidean space.

(a) By integrating out the Fourier modes of $\phi(\mathbf{r})$ with wavevectors $\Lambda e^{-l} < |\mathbf{q}| < \Lambda$, implement the momentum-shell renormalisation group procedure to first order in u and derive the following flow equations

$$\begin{aligned} \frac{dK}{dl} &= \frac{4u\Lambda^2}{K\pi}, \\ \frac{du}{dl} &= -2u. \end{aligned}$$

Let

$$G(r, K, u) \equiv \langle e^{i\phi(\mathbf{r}) - i\phi(\mathbf{0})} \rangle_{\beta\mathcal{H}},$$

where the expectation value with respect to the above Hamiltonian depends on the parameters (K, u).

(b) Show that

$$G(r, K, u = 0) = \frac{1}{(r/a)^{\frac{1}{2\pi K}}},$$

where a is the lattice constant.

(c) Show that an RG trajectory starting at the point (K_0, u_0) in the (K, u) plane flows towards the point $(\sqrt{K_0^2 + 4u_0\Lambda^2/\pi}, 0)$.

(d) Considering an infinitesimal RG flow from l to $l + \delta l$ starting at the point $(K \gg 1, u)$, show that

$$G(r, K, u) = e^{-\frac{\delta l}{2\pi K}} G\left(r(1 - \delta l), K + \frac{dK}{dl}\delta l, u + \frac{du}{dl}\delta l \right). \tag{1}$$

(e) Now, consider a series of infinitesimal RG flows from $l = 0$ to $l = \ln \frac{r_0}{r}$, starting at the point (K_0, u_0) and ending at the point $\left(K\left(\ln \frac{r_0}{r}\right), u\left(\ln \frac{r_0}{r}\right)\right)$, to show that

$$G(r_0, K_0, u_0) = \exp\left(-\int_0^{\ln(r_0/r)} \frac{dl}{2\pi K(l)}\right) G\left(r, K\left(\ln \frac{r_0}{r}\right), u\left(\ln \frac{r_0}{r}\right)\right). \qquad (2)$$

(f) Hence, show that the asymptotic limit of the correlator is given by

$$G(r_0, K_0, u_0) \overset{r_0/r\to\infty}{=} (1 + \mathcal{O}(u_0)) \frac{1}{(r_0/a)^{\frac{1}{2\pi K_*}}},$$

i.e. the long-distance physics is given by the quadratic theory, but with a renormalised coupling constant $K_* = \sqrt{K_0^2 + 4u_0\Lambda^2/\pi}$.

The next problem concerns the ϵ-expansion of the Ginzburg-Landau Hamiltonian to second order. Although outlined in the lectures, this problem leads you through a detailed investigation of the O(n) fixed point. In attacking this problem one may wish to consult a reference text such as Chaikin and Lubensky (p. 263).

2. Using Wilson's perturbative renormalisation group, the aim of this problem is to obtain the second-order $\epsilon = 4 - d$ expansion of the Ginzburg-Landau functional

$$\beta H = \int d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + \frac{K}{2}(\nabla \mathbf{m})^2 + u(\mathbf{m}^2)^2\right],$$

where $\mathbf{m}$ denotes an n-component field.

(a) Treating the quartic interaction as a perturbation, show that an application of the momentum shell RG generates a Hamiltonian of the form

$$\beta H[\mathbf{m}_<] = \int_0^{\Lambda/b} (d\mathbf{q}) \frac{G^{-1}(\mathbf{q})}{2} |\mathbf{m}_<(\mathbf{q})|^2 - \ln\left\langle e^{-U}\right\rangle_{\mathbf{m}_>}, \qquad G^{-1}(\mathbf{q}) = t + K\mathbf{q}^2,$$

where we have used the shorthand $(d\mathbf{q}) \equiv d\mathbf{q}/(2\pi)^d$.

(b) Expressing the interaction in terms of the Fourier modes of the Gaussian Hamiltonian, represent *diagrammatically* those contributions from the second order of the cummulant expansion. [Remember that the cummulant expansion involves only those diagrams which are *connected*.]

(c) *Focusing only on those second order contributions that renormalise the quartic interaction*, show that the renormalised coefficient u takes the form

$$\widetilde{u} \;=\; u - 4u^2(n+8)\int_{\Lambda/b}^{\Lambda} (d\mathbf{q}) G(\mathbf{q})^2.$$

Comment on the nature of those additional terms generated at second-order.

(d) Applying the rescaling $\mathbf{q} = \mathbf{q}'/b$, performing the renormalisation $\mathbf{m}_< = z\mathbf{m}$, and arranging that $K' = K$, show that the differential recursion relations take the form ($b = e^\ell$)

$$\begin{aligned}\frac{dt}{d\ell} &= 2t + 4u(n+2)G(\Lambda)K_d\Lambda^d - u^2 A(\mathbf{q}=0),\\ \frac{du}{d\ell} &= (4-d)u - 4(n+8)u^2 G(\Lambda)^2 K_d \Lambda^d.\end{aligned}$$

(e) From this result, show that for $d < 4$ the Gaussian fixed point becomes unstable against a new fixed point (known as the O(n) fixed point). [Remember to be consistent in keeping terms of definite order in ϵ!] Linearising in the vicinity of the new fixed point, show that the scaling dimensions take the form

$$y_t = 2 - \left(\frac{n+2}{n+8}\right)\epsilon + O(\epsilon^2), \qquad y_u = -\epsilon + O(\epsilon^2).$$

Sketch the RG flows for $d > 4$ and $d < 4$.

(f) Adding the magnetic field dependent part of the Hamiltonian, show that to leading order in ϵ, the magnetic exponent y_h is unchanged from the mean-field value.

(g) From the scaling relations for the free energy density and correlation length

$$\begin{aligned}f(g_1 = \delta t, h) &= b^{-d} f(b^{y_t}\delta t, b^{y_h} h),\\ \xi(\delta t, h) &= b^{-1}\xi(b^{y_t}\delta t, b^{y_h} h).\end{aligned}$$

determine the critical exponents ν, α, β, and γ. [Recall: $\xi \sim (\delta t)^{-\nu}$, $C \sim (\delta t)^{-\alpha}$, $m \sim (\delta t)^{\beta}$, $\chi \sim (\delta t)^{-\gamma}$.]

Optional Problem for Enthusiasts: The final problem in this set is *optional* and involves another investigation of an ϵ-expansion this time applied to continuous spins near two-dimensions. In contrast to the $4-\epsilon$ expansion of the Ginzburg-Landau Hamiltonian described above, a non-trivial fixed point emerges already at first order. The aim of this calculation is to study properties of the fixed point in the vicinity of two-dimensions. This calculation repeats steps first performed by Polyakov (Phys. Lett. **59B**, 79 (1975)) in a seminal work on the properties of the non-linear σ-model. Once again, this calculation should be attempted with reference to a standard text such as Chaikin and Lubensky (p. 341).

3. ∗∗ *Optional Question on Continuous Spin Systems Near Two-Dimensions*: The aim of this problem is to employ Wilson's perturbative renormalisation group, to obtain the $\epsilon = d - 2$ expansion of the n-component non-linear σ-model

$$\mathcal{Z} = \int D\mathbf{S}(\mathbf{x})\delta\left(\mathbf{S}^2(\mathbf{x}) - 1\right)\exp\left[-\frac{K}{2}\int d\mathbf{x}\,(\nabla\mathbf{S})^2\right].$$

In the vicinity of the transition temperature, it is convenient to expand the spin degrees of freedom around the (arbitrary) direction of spontaneous symmetry breaking, $\mathbf{S}_0(\mathbf{x}) = (0, \cdots 0, 1)$,

$$\mathbf{S}(\mathbf{x}) = (\Pi_1(\mathbf{x}), \cdots \Pi_{n-1}(\mathbf{x}), \sigma(\mathbf{x})) \equiv (\Pi(\mathbf{x}), \sigma(\mathbf{x})),$$

where $\sigma(\mathbf{x}) = (1 - \Pi^2)^{1/2}$.

(i) Substituting this expression, and expanding σ in powers of Π, show that the Hamiltonian takes the form

$$\beta H = \frac{K}{2}\int d\mathbf{x}\left[(\nabla\Pi)^2 + \frac{1}{2}\left(\nabla\Pi^2\right)^2 + \cdots\right].$$

(ii) Treating this expansion to quadratic order, show that the lower critical dimension is 2.

(iii) Taking $\sigma > 0$, and using the expression (true when $\sigma > 0$)

$$\delta\left(\Pi^2 + \sigma^2 - 1\right) = \frac{1}{2(1 - \Pi^2)^{1/2}}\delta\left(\sigma - (1 - \Pi^2)^{1/2}\right),$$

show that the partition function can be written in the form

$$\mathcal{Z} = \int D\Pi(\mathbf{x})\exp\left[-\frac{\rho}{2}\int d\mathbf{x}\ln(1 - \Pi^2)\right] \times \exp\left\{-\frac{K}{2}\int d\mathbf{x}\left[(\nabla\Pi)^2 + \left(\nabla(1 - \Pi^2)^{1/2}\right)^2\right]\right\},$$

where $\rho \equiv (N/V) = \int_0^\Lambda (d\mathbf{q})$ denotes the density of states.

(iv) *Polyakov's Perturbative Renormalisation Group*: Expanding the Hamiltonian perturbatively in Π, show that $K\langle\Pi^2\rangle \sim O(1)$, $K(\nabla\Pi^2)^2 \sim O(K^{-1})$, and $\rho\Pi^2 \sim O(K^{-1})$.

This suggests that we define

$$\beta H_0 = \frac{K}{2}\int d\mathbf{x}\,(\nabla\Pi)^2,$$

as the unperturbed Hamiltonian and treat

$$U = \frac{K}{2}\int d\mathbf{x}\,(\Pi\cdot\nabla\Pi)^2 - \frac{\rho}{2}\int d\mathbf{x}\,\Pi^2,$$

as a perturbation.

(v) Expand the interaction in terms of the Fourier modes and obtain an expression for the propagator $\langle \Pi_\alpha(\mathbf{q}_1)\Pi_\beta(\mathbf{q}_2)\rangle_0$. Sketch a diagrammatic representation of the components of the perturbation.

(vi) *Perturbative Renormalisation Group*: Applying the perturbative RG procedure, and integrating out the fast degrees of freedom, show that the partition function takes the form

$$\mathcal{Z} = \int D\Pi_< e^{-\delta f_b^0 - \beta H_0[\Pi_<] - \ln\left\langle e^{-U[\Pi_<,\Pi_>]}\right\rangle_0^>},$$

where δf_b^0 represents some constant.

(vii) Expanding to first order, identify and obtain an expression for the *two* diagrams that contribute towards a renormalisation of the coupling constants. (Others either vanish or give a constant contribution.) [Note: the density of states is given by $\rho = (N/V) = \int_0^\Lambda (d\mathbf{q}) = b^d \int_0^{\Lambda/b}(d\mathbf{q})$.] As a result, show that the renormalised Hamiltonian takes the form

$$\begin{aligned} -\beta H[\Pi_<] &= \delta f_b^0 + \delta f_b^1 - \frac{\widetilde{K}}{2}\int_0^{\Lambda/b} d\mathbf{x}(\nabla\Pi_<)^2 + \frac{\rho}{2}b^{-d}\int_0^{\Lambda/b} d\mathbf{x}|\Pi_<|^2 \\ &\quad -\frac{K}{2}\int_0^{\Lambda/b} d\mathbf{x}\left(\Pi_{<\alpha}\nabla\Pi_{<\alpha}\right)^2 + O(K^{-2}), \end{aligned}$$

where $\widetilde{K} = K(1 + I_d(b)/K)$ and δf_b^0, δf_b^1 are constants. Specify the function $I_d(b)$.

(viii) Applying the rescaling $\mathbf{x}' = \mathbf{x}/b$ and renormalising the spins,

$$\mathbf{S}' = \frac{\mathbf{S}}{\zeta}, \qquad \Pi_< = \zeta\Pi',$$

obtain an expression for the renormalised coupling constant K'.

To determine ζ, it is necessary to evaluate the average of the renormalised spin $\langle\mathbf{S}\rangle_0 = \langle(\Pi_{<1} + \Pi_{>1}, \cdots (1 - \Pi_<^2 - \Pi_>^2)^{1/2})\rangle_0$. Expanding, we find

$$\begin{aligned} \langle\mathbf{S}\rangle_0^> &= \left(\Pi_{<1}, \cdots, 1 - \Pi_<^2/2 - \left\langle\Pi_>^2\right\rangle/2\right) \\ &\approx \left(1 - \left\langle\Pi_>^2\right\rangle/2\right)\left(\Pi_{<1}, \cdots, 1 - \Pi_<^2/2\right) = \zeta\mathbf{S}' \end{aligned}$$

From this expression, show that $\zeta = 1 - (n-1)I_d(b)/2K$.

(ix) Using the expression for K' and ζ, show that the differential recursion relation takes the form

$$\frac{dK}{d\ell} = (d-2)K - (n-2)K_d\Lambda^{d-2},$$

where $b = e^\ell$. Setting the temperature $T = K^{-1}$, obtain the recursion relation $dT/d\ell$ and confirm that the fixed point is given by

$$T^* = \frac{d-2}{(n-2)K_d\Lambda^{d-2}} = \frac{2\pi\epsilon}{(n-2)} + O(\epsilon^2),$$

where $d = 2+\epsilon$. Sketch the RG flow diagram for $d > 2$, $d = 2$ and $d < 2$, for various values of n.

(x) Linearising the RG flow in the vicinity of the fixed point, obtain the thermal exponent y_t to leading order in ϵ. Using this result, obtain the correlation length exponent $\nu = 1/y_t$.

(xi) Adding a term $-\int d\mathbf{x}\mathbf{h}\cdot\mathbf{S}$ show that the magnetic exponent takes the form

$$y_h = 2 + \frac{n-3}{2(n-2)}\epsilon + O(\epsilon^2).$$

(xii) Using an exponent identity, obtain the critical exponent γ. Setting $d = 3$ and $n = 3$, how does this estimate compare to the best estimate of 1.38.