Chapter 5

Topological Phase Transitions

Previously, we have seen that the breaking of a continuous symmetry is accompanied by the appearance of massless Goldstone modes. Fluctuations of the latter lead to the destruction of long-range order at any finite temperature in dimensions $d \leq 2$ — the Mermin-Wagner theorem. However, our perturbative analysis revealed only a power-law decay of spatial correlations in precisely two-dimensions — "quasi long-range order". Such cases admit the existence of a new type of continuous phase transition driven by the proliferation of topological defects. The aim of this section is to discuss the phenomenology of this type of transition which lies outside the usual classification scheme.

In classifying states of condensed matter, we usually consider two extremes: on the one hand there are crystalline solids in which atoms form a perfectly periodic array that extends to infinity in all directions. Such phases are said to possess **long-range order** (LRO). On the other hand there are fluids or glasses, in which the atoms are completely disordered and the system is both orientationally and positionally isotropic — that is the materials look the same when viewed from any direction. However, an intermediate state of matter is possible. In such a state the atoms are distributed at random, as in a fluid or glass, but the system is orientationally anisotropic on a macroscopic scale, as in a crystalline solid. This means that some properties of the fluid are different in different directions. Order of this sort is known as **bond-orientational order**.

This type of **quasi long-range order** is manifest in properties of superfluid and superconducting films (i.e. two-dimensions) and in the crystallisation properties of fluid membranes. As we have seen, according to the Mermin-Wagner theorem, fluctuations of a two-component or complex order parameter destroy LRO at all finite temperatures. However, at temperatures below T_c, quasi-LRO is maintained. The nature of this **topological phase transition** was first resolved by Berezinskii (Sov. Phys. JETP **32**, 493, (1971)) and later generalised to encompass a whole class of systems by Kosterlitz and Thouless[1] (J. Phys. C **5**, L124 (1972); **6**, 1181 (1973)). These include the melting of a two-dimensional crystal, with dislocations taking the place of vortices (Halperin and Nelson, Phys. Rev. Lett. **41**, 121 (1978)).

[1]

In this chapter, we will exploit a magnetic analogy to explore this unconventional type of phase transition which is driven by the condensation of **topological defects** known as vortices. Note that this type of phase transition is qualitatively quite different from those we have met previously.

5.1 Continuous Spins Near Two-Dimensions

Suppose unit n-component spins $\mathbf{S}_i = (s_{i1}, s_{i2}, \cdots s_{in})$ ($\mathbf{S}_i^2 = 1$) which occupy the sites i of a lattice and interact ferromagnetically with their neighbours.

$$-\beta H = K \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -\frac{K}{2} \sum_{\langle ij \rangle} \left[(\mathbf{S}_i - \mathbf{S}_j)^2 - 2 \right].$$

5.1.1 High Temperature Series

As usual we can try to confirm the existence of two separate phases, at high and low temperatures, by respectively treating β as a small or large parameter in the partition function. In the former case, we can expand the exponential in the partition function as follows

$$\mathcal{Z} = \int D\mathbf{S}_i\, \delta(\mathbf{S}^2 - 1)\, e^{-\beta H} = \int D\mathbf{S}_i\, \delta(\mathbf{S}^2 - 1) \left[1 + K \sum_{<ij>} S_i^\mu S_j^\mu + O(K^2) \right],$$

where we have used the notation $\delta(\mathbf{S}^2 - 1)$ to represent a "functional δ-function" — i.e., at all spatial coordinates, $\mathbf{S}(\mathbf{x})^2 = 1$. Summation over repeated Cartesian components μ is implied.

The high temperature expansion can be used to estimate the spin-spin correlation function $\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle$. The terms in the high temperature series are products of factors. Each factor in a given product corresponds to a lattice bond $< ij >$. To leading order, only those products with factors which join sites 0 and $\mathbf{r}$ will survive and give a contribution. This is because once the integral over $\mathbf{S}_i$ is taken we have $\langle S_i^\mu S_j^\nu \rangle = \frac{1}{n} \delta_{\mu\nu} \delta_{ij}$, where the average is taken with respect to all possible configurations of $\mathbf{S}_i$.

$$\boxed{\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle \sim \left(\frac{K}{n} \right)^{|\mathbf{x}|} \sim \exp\left[-|\mathbf{x}|/\xi \right]}$$


John Michael Kosterlitz and David James Thouless: together with Duncan Haldane co-recipients of the 2016 Nobel Prize in Physics "for theoretical discoveries of topological phase transitions and topological phases of matter".

where $\xi^{-1} = \ln(n/K)$. This result implies an *exponential decay* of the spin-spin correlation function in the *disordered phase*. Note that that the number of possible lattice paths that connect two points can scale with at most $d^{|\mathbf{x}|}$ – we have neglected these non-universal lattice effects.

5.1.2 Low Temperature Series

At zero temperature the presumption is that the ground state configuration is ferromagnetic with all spins aligned along some direction (say $\mathbf{S}_i = \hat{\mathbf{e}}_n \equiv (0, 0, \cdots, 1)$). At low temperatures statistical fluctuations involve only low energy long wavelength modes which can be treated within a continuum approximation. Accordingly the Hamiltonian can be replaced by

$$\boxed{-\beta H[\mathbf{S}] = -\beta E_0 - \frac{K}{2}\int d\mathbf{x}\ (\nabla \mathbf{S})^2,}$$

where the discrete lattice index i has been replaced by a continuous vector $\mathbf{x} \in R^d$. The corresponding partition function is given by the so-called **non-linear σ-model**,

$$\boxed{\mathcal{Z} = \int D\mathbf{S}(\mathbf{x})\ \delta(\mathbf{S}^2 - 1)\ e^{-\beta H[\mathbf{S}]}.}$$

Here we have used the notation $\delta(\mathbf{S}^2 - 1)$ to represent a "functional δ-function" — i.e. at all spatial coordinates, $\mathbf{S}(\mathbf{x})^2 = 1$.

Fluctuations transverse to the ground state spin orientation $\hat{\mathbf{e}}_n$ are described by $n-1$ **Goldstone modes**. Adopting the parameterisation $\mathbf{S}(\mathbf{x}) = (\pi_1(\mathbf{x}), \cdots \pi_{n-1}(\mathbf{x}), (1 - \pi^2)^{1/2}) \equiv (\pi, (1-\pi^2)^{1/2})$, and expanding to quadratic order in π we obtain the following expression for the average transverse fluctuation (cf. section 2.5)

$$\begin{aligned}\langle|\pi(\mathbf{x})|^2\rangle &= \int \frac{d^d\mathbf{q}}{(2\pi)^d}\langle|\pi(\mathbf{q})|^2\rangle = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{n-1}{K\mathbf{q}^2} \\ &= \frac{n-1}{K}\frac{S_d}{(2\pi)^d}\frac{a^{2-d} - L^{2-d}}{d-2} \xrightarrow{L\to\infty} \frac{(n-1)K_d}{K}\begin{cases} a^{2-d} \propto T & d > 2, \\ L^{2-d} \to \infty & d \leq 2.\end{cases}\end{aligned}$$

This result suggests that in more than two dimensions we can always find a temperature where the magnitude of the fluctuations is small while in dimensions of two or less fluctuations always destroy long-range order. This is in accord with the Mermin-Wagner theorem discussed in section 2.5 which predicted the absence of long-range order in $d \leq 2$. Even so, for $d = 2$ the low temperature analysis still indicates the presence of a low-temperature phase which is distinct from the high-temperature phase with a finite correlation length. This phase, rather than exhibiting true long-range order has **quasi long-range order** (power-law order)

$$\langle \mathbf{S}(\mathbf{0}) \cdot \mathbf{S}(\mathbf{x})\rangle \approx e^{-\frac{n-1}{2\pi K}\ln\left(\frac{\mathbf{x}}{a}\right)} = \left(\frac{a}{|\mathbf{x}|}\right)^{\frac{n-1}{2\pi K}} \tag{5.1}$$

This analysis is in fact incorrect for $n > 2$ as higher order interaction terms between the $n-1$ Goldstone mode branches are relevant. For $n = 2$ there is only one Goldstone mode branch and the low temperature expansion is stable – the power-law phase extends to finite temperatures. The mechanism behind the phase transition responsible for loss of power-law order at high temperatures will be the subject of the next section. We now demonstrate in the case of $n = 3$ that interactions between two different Goldstone mode branches are relevant. (This argument carries through to $n > 3$).

$$
\begin{aligned}
\beta H[\mathbf{S}(\mathbf{x})] &= \frac{K}{2}\int d^d\mathbf{x}\,(\mathbf{e}_\theta\partial_\mu\theta + \mathbf{e}_\phi\sin\theta\partial_\mu\phi)\cdot(\mathbf{e}_\theta\partial_\mu\theta + \mathbf{e}_\phi\sin\theta\partial_\mu\phi) \\
&\overset{\substack{\theta\to\frac{\pi}{2}+\theta \\ K=T^{-1}}}{=} \frac{1}{2T}\int d^d\mathbf{x}\left[\partial_\mu\theta\partial^\mu\theta + \partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\theta^2\partial_\mu\phi\partial^\mu\phi + \mathcal{O}(\theta^4\phi^2)\right] \\
&\overset{\substack{\theta\to\sqrt{T}\theta \\ \phi\to\sqrt{T}\phi}}{=} \int d^d\mathbf{x}\left[\partial_\mu\theta\partial^\mu\theta + \partial_\mu\phi\partial^\mu\phi - \frac{T}{2}\theta^2\partial_\mu\phi\partial^\mu\phi + \mathcal{O}(\theta^4\phi^2)\right]
\end{aligned}
\tag{5.2}
$$

where $\theta(\mathbf{r})$ and $\phi(\mathbf{r})$ are the spherical polar angles of the spin vector $\mathbf{S}(\mathbf{r})$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ the corresponding unit direction vectors, and $(\theta = \pi/2, \phi = 0)$ is the globally uniform configuration around which we are expanding. We have also explicitly shown that the quartic interaction term is small in T. This is why T is referred to as the **coupling** as it is a measure of the interaction strength between the Goldstone modes. Under naive RG scaling of $(\theta,\phi) \to b^{\frac{2-d}{2}}(\theta,\phi)$ and $\mathbf{x} \to b\mathbf{x}$, it is clear that this term is relevant in $d \leq 2$. In fact, Polyakov [Phys. Lett. **59**B, 79 (1975)] developed a perturbative RG expansion close to two-dimensions that shows that the interactions between these Goldstone modes lead to the instability of the low-temperature fixed point for $d \leq 2$, i.e., the system flows towards the high-temperature $K = 0$ fixed point as soon as K becomes finite.[2]

The excitation of Goldstone modes therefore rules out spontaneous order in two-dimensional models with a continuous symmetry. An RG analysis of the non-linear σ-model indeed confirms that the transition temperature of n-component spins vanishes as $T^* = 2\pi\epsilon/(n-2)$ for $\epsilon = (d-2) \to 0$ (see problem set 2). This unstable fixed point that separates the low and high temperature phases moves to a finite temperature as d is increased above 2. The RG also indicates that the behaviour for $n = 2$ is in some sense marginal.

RG flow for $n = 3$ **and** $d = 2$: We now analyse the RG flow equation, derived by Polyakov, in more detail in the case of $n = 3$ and $d = 2$ (see problem set 2)

$$
\boxed{\frac{dT}{dl} = \frac{T^2}{2\pi},}
\tag{5.3}
$$

[2]Polyakov's work provided one of the milestones in the study of critical phenomena. The $\epsilon = d-2$ expansion employed in the perturbative RG approach set the framework for numerous subsequent investigations. A description of the RG calculation can be found in Chaikin and Lubensky and is assigned as a question in the problem set 2.

where $T = K^{-1}$. Integrating the above equation we obtain

$$\frac{1}{T} - \frac{1}{T'} = \frac{l}{2\pi}, \tag{5.4}$$

where $T' = T$ at $l = 0$. We can also write down separate flow equations for the correlation length ξ and the momentum p

$$\begin{aligned} \xi' &= \xi e^{-l}, \\ p' &= p e^{l}. \end{aligned} \tag{5.5}$$

If we choose $T \sim \mathcal{O}(1)$, then $\xi \sim a$. We thus obtain

$$\boxed{\xi' \sim a e^{\frac{2\pi}{T'}},} \tag{5.6}$$

for the divergence of the correlation length as $T' \to 0$. This divergence is non-perturbative, i.e., it could not have been obtained from any finite order of perturbation theory.

Running coupling: In general, we can map correlation functions at momentum p and coupling T to ones at momentum p' and coupling T'. The change of correlators with l can be obtained from the RG flow equations and is described by the Callan-Symanzik equation. We will demonstrate this idea by considering the flow of the following non-linear σ model correlator for the case $n = 3$, $d = 2$

$$G(x, T) = \langle \mathbf{S}(\mathbf{x}) \cdot \mathbf{S}(0) \rangle_T \approx \langle 1 + 2\phi(\mathbf{x})\phi(0) - 2\phi^2(0) \rangle_T, \tag{5.7}$$

where $\phi(\mathbf{x})$ is the azimuthal angle of the three-component spin and to leading order its renormalisation factor $\zeta = 1$ (see problem set 3). Note that, to leading order, the fast and slow parts of $\phi(\mathbf{x})$ separate. Neglecting the quartic terms, we can write down

$$\langle 1 + 2\phi(\mathbf{x})\phi(0) - 2\phi^2(0) \rangle_T \approx \langle 1 + 2\phi_<(\mathbf{x})\phi_<(0) - 2\phi_<^2(0) \rangle \langle 1 + 2\phi_>(\mathbf{x})\phi_>(0) - 2\phi_>^2(0) \rangle \,. \tag{5.8}$$

Considering the flow of the correlator from $l = 0$ to δl, we obtain for $\delta l \ll 1$

$$\begin{aligned} G(x, T) &= G\left(xe^{-\delta l}, T'(\delta l)\right)\left(1 - 2\langle \phi^2(0) - \phi(\mathbf{x})\phi(0) \rangle_>\right) \\ &= G\left(xe^{-\delta l}, T'(\delta l)\right)\left(1 - 2\int_{\Lambda e^{-\delta l}}^{\Lambda} \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{T}{\mathbf{q}^2}\left(1 - e^{i\mathbf{q}\cdot\mathbf{x}}\right)\right) \\ &\overset{\Lambda|\mathbf{x}|\gg 1}{=} G\left(xe^{-\delta l}, T'(\delta l)\right)\left(1 - 2\int_{\Lambda e^{-\delta l}}^{\Lambda} \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{T}{\mathbf{q}^2}\right) \\ &= G\left(xe^{-\delta l}, T'(\delta l)\right) e^{-\frac{T}{\pi}\delta l}, \end{aligned} \tag{5.9}$$

where the factor is the result of integrating out fluctuations $\Lambda e^{-l} \leq |\mathbf{q}| \leq \Lambda$, which are not present in the correlator that is evaluated with the renormalised parameters. Taking the Fourier transform of both sides of the equation, we obtain

$$G(p, T) = e^{-\frac{T}{\pi}\delta l + 2\delta l} G(pe^{\delta l}, T'(\delta l)) \,. \tag{5.10}$$

Considering a succession of RG transformations we thus obtain

$$G(p,T) = e^{2l}\Gamma(l)G(pe^{l},T'(l)), \tag{5.11}$$

where Γ is the amplitude factor accumulated through a series of integrations over the short-distance fluctuations

$$\Gamma(l) = \exp\left[-\int_0^l \frac{T'(l)}{\pi}dl\right], \tag{5.12}$$

and the integrand in the exponential $\gamma(T) = -\frac{T}{\pi}$ is known as the **gamma function**.

Choosing an l such that $pe^l = p'$, we obtain

$$G(p,T) = \Gamma\left(\ln\frac{p'}{p}\right)\left(\frac{p'}{p}\right)^2 G\left(p', T'\left(\ln\frac{p'}{p}\right)\right), \tag{5.13}$$

where

$$\boxed{T'\left(\ln\frac{p'}{p}\right) = \frac{1}{\frac{1}{T}+\frac{1}{2\pi}\ln\frac{p}{p'}} \rightarrow \frac{2\pi}{\ln\frac{p}{p'}} \quad \text{as} \quad \frac{p}{p'}\rightarrow\infty} \tag{5.14}$$

is known as the running (or effective) coupling constant at momentum p and its derivative $\frac{dT'}{d\ln\frac{p}{p'}} = \beta(T')$ is known as the **beta function**. Hence, in the limit $p/p' \rightarrow \infty$ ($p' \sim \mathcal{O}(1)$ is kept fixed and the lattice cutoff $1/a$ has been taken to infinity), we can expand $G\left(p', T'\left(\ln\frac{p'}{p}\right)\right)$ in small T'. In the limit $T \rightarrow 0$ the correlation length ξ diverges and the correlator $G(x, T\rightarrow 0) \rightarrow \frac{1}{x^{T/\pi}}$ at a fixed lengthscale $x \ll \xi$, and its Fourier transform is therefore $G(p,T) \rightarrow 1/p^{2-T/\pi}$

$$G\left(p', T'\left(\ln\frac{p'}{p}\right)\right) = \frac{1}{p'^2} + \mathcal{O}\left(\frac{1}{\ln p/p'}\right), \tag{5.15}$$

$$G(p,T) = \Gamma\left(\ln\frac{p'}{p}\right)\left[\frac{1}{p^2} + \mathcal{O}\left(\frac{1}{\ln p/p'}\right)\right]. \tag{5.16}$$

The correlators tend to those of a purely quadratic theory in the large momentum limit with logarithmically small corrections. In other words, the effective (or **running**) coupling becomes logarithmically small at large momenta.

Exercise for the Reader:
Show that the gamma function gives rise to logarithmic corrections
$\Gamma\left(\ln\frac{p'}{p}\right) \propto \ln^2\frac{p'}{p}$ as $\frac{p'}{p} \rightarrow \infty$.

5.2 Topological Defects in the XY-Model

The first indication of unusual behaviour in the two-dimensional XY-model ($n = 2$) appeared in an analysis of the high temperature series expansion by Stanley and Kaplan (1971). The series appeared to indicate a divergence of susceptibility at a finite temperature, seemingly in contradiction with the absence of symmetry breaking. It was this contradiction that led Wigner to explore the possibility of a *phase transition without symmetry breaking*. It is the study of this novel and important type of phase transition to which we now turn. We begin our analysis with a study of the asymptotic behaviour of the partition function at high and low temperatures using a series expansion.

5.2.1 High Temperature Series

In two-dimensions it is convenient to parameterise the spins by their angle with respect to the direction of one of the ground state configurations $\mathbf{S} = (\cos\theta, \sin\theta)$. The spin Hamiltonian can then be presented in the form

$$\boxed{-\beta H = K\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j).}$$

At high temperatures the partition function can be expanded in powers of K

$$\mathcal{Z} = \int_0^{2\pi}\prod_i \frac{d\theta_i}{2\pi}e^{-\beta H} = \int_0^{2\pi}\prod_i \frac{d\theta_i}{2\pi}\prod_{\langle ij\rangle}\left[1 + K\cos(\theta_i - \theta_j) + O(K^2)\right].$$

Each term in the product can be represented by a "bond" that connects neighbouring sites i and j. To the lowest order in K, each bond on the lattice contributes either a factor of one, or $K\cos(\theta_i - \theta_j)$. But, since $\int_0^{2\pi}(d\theta_1/2\pi)\cos(\theta_1 - \theta_2) = 0$ any graph with a single bond emanating from a site vanishes. On the other hand, a site at which two-bonds meet yields a factor $\int_0^{2\pi}(d\theta_2/2\pi)\cos(\theta_1 - \theta_2)\cos(\theta_2 - \theta_3) = \cos(\theta_1 - \theta_3)/2$. The first non-vanishing contributions to the partition function arise from closed loop configurations that encircle one plaquette.

The high temperature expansion can be used to estimate the spin-spin correlation function $\langle \mathbf{S}_0 \cdot \mathbf{S}_{\mathbf{x}}\rangle = \langle\cos(\theta_{\mathbf{x}} - \theta_0)\rangle$. To leading order, only those graphs which join sites 0 and $\mathbf{r}$ will survive and give a contribution

$$\boxed{\langle \mathbf{S}_0 \cdot \mathbf{S}_{\mathbf{x}}\rangle \sim \left(\frac{K}{2}\right)^{|\mathbf{x}|} \sim \exp\left[-|\mathbf{x}|/\xi\right],}$$

where $\xi^{-1} = \ln(2/K)$. This result implies an *exponential decay* of the spin-spin correlation function in the *disordered phase*.

5.2.2 Low Temperature Series

At low temperature the cost of small fluctuations around the ground state is obtained within a quadratic expansion which yields the Hamiltonian corresponding to Eq. (2.18)

$$-\beta H = \frac{K}{2}\int d\mathbf{x}(\nabla\theta)^2,$$

in the continuum limit. Note that the integration measure $d^2\mathbf{x}$ is in units of a. According to the standard rules of Gaussian integration

$$\langle \mathbf{S}(0)\cdot\mathbf{S}(\mathbf{x})\rangle = \mathrm{Re}\left\langle e^{i(\theta(0)-\theta(\mathbf{x}))}\right\rangle = \mathrm{Re}\left[e^{-\langle(\theta(0)-\theta(\mathbf{x}))^2\rangle/2}\right].$$

In section 2.5 we saw that in two-dimensions Gaussian fluctuations grow logarithmically $\langle(\theta(0)-\theta(\mathbf{x}))^2\rangle/2 = \ln(|\mathbf{x}|/a)/2\pi K$, where a denotes a short distance cut-off (i.e. lattice spacing). Therefore, at low temperatures the spin-spin correlation function decays *algebraically* as opposed to exponential.

$$\boxed{\langle \mathbf{S}(0)\cdot\mathbf{S}(\mathbf{x})\rangle \simeq \left(\frac{a}{|\mathbf{x}|}\right)^{1/2\pi K}.}$$

A power law decay of correlations implies self-similarity (i.e. no correlation length), as is usually associated with a critical point. Here it arises from the logarithmic growth of angular fluctuations, which is specific to two-dimensions.

The distinction between the nature of the asymptotic decays allows for the possibility of a finite temperature phase transition. However, the arguments so far put forward are not specific to the XY-model. Any continuous spin model in $d = 2$ will exhibit exponential decay of correlations at high temperature, and a power law decay in a low temperature Gaussian approximation. Strictly speaking, to show that Gaussian behaviour persists to low temperatures we must prove that it is not modified by the additional terms in the gradient expansion. Quartic terms, such as $\int d^d\mathbf{x}(\nabla\theta)^4$, generate interactions between Goldstone modes belonging to the same branch and naive RG scaling suggests they are irrelevant in $d = 2$. This can be confirmed using perturbative RG (see problem set 2).

Exercise for the Reader:
Show that naive RG scaling suggests that terms $\int d^d\mathbf{x}(\nabla\theta)^4$ are irrelevant.

We have already seen that the zero temperature fixed point in $d = 2$ is unstable for all $n > 2$ but apparently stable for $n = 2$. (There is only one branch of Goldstone modes for $n = 2$. It is the interactions between different branches of these modes for $n > 2$ that are relevant and lead to instability towards high temperature behaviour.) The low temperature phase of the XY-model is said to possess **quasi-long range order**, as opposed to **true long range order** that accompanies finite magnetisation.

What is the mechanism for the disordering of the quasi-long range ordered phase? Since the RG suggests that higher order terms in the gradient expansion are not relevant it is necessary to search for other relevant operators.

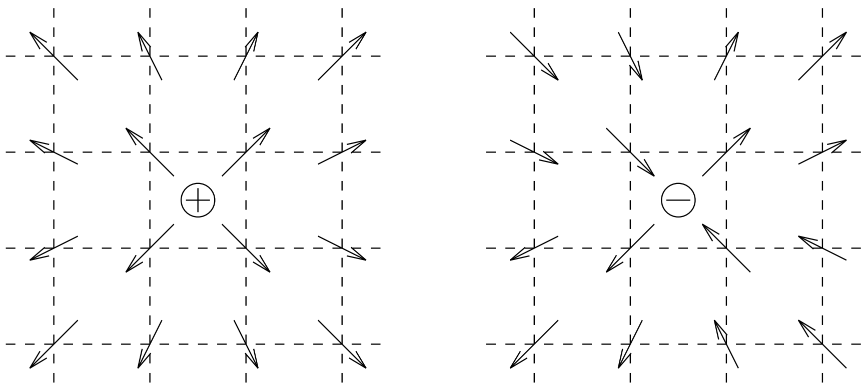

Figure 5.1: Spin configurations of the two-dimensional XY-model showing vortices of charge ± 1.

5.3 Vortices

The gradient expansion describes the energy cost of *small* deformations around the ground state and applies to configurations that can be continuously deformed to the uniformly ordered state. Berezinskii, and later Kosterlitz and Thouless, suggested that the disordering is caused by **topological defects** that can not be obtained from such deformations.

Since the angle describing the orientation of a spin is defined up to an integer multiple of 2π, it is possible to construct spin configurations in which the traversal of a closed path will see the angle rotate by $2\pi n$. The integer n is the **topological charge** enclosed by the path. The discrete nature of the charge makes it impossible to find a continuous deformation which returns the state to the uniformly ordered configuration in which the charge is zero. (More generally, topological defects arise in any model with a compact group describing the order parameter — e.g. a 'skyrmion in an $O(3)$' or three-component spin Heisenberg Ferromagnet, or a dislocation in a crystal.)

The elementary defect, or **vortex**, has a unit charge. In completing a circle centred on the defect the orientation of the spin changes by $\pm 2\pi$ (see Fig. 5.1). If the radius r of the circle is sufficiently large, the variations in angle will be small and the lattice structure can be ignored. By symmetry $\nabla\theta$ has uniform magnitude and points along the azimuthal direction. The magnitude of the distortion is obtained by integrating around a path that encloses the defect,

$$\boxed{\oint \nabla\theta \cdot d\ell = 2\pi n \quad \Longrightarrow \quad \nabla\theta = \frac{n}{r}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z,}$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$ are unit vectors respectively in the plane and perpendicular to it. This (continuum) approximation fails close to the centre (core) of the vortex, where the lattice structure is important.

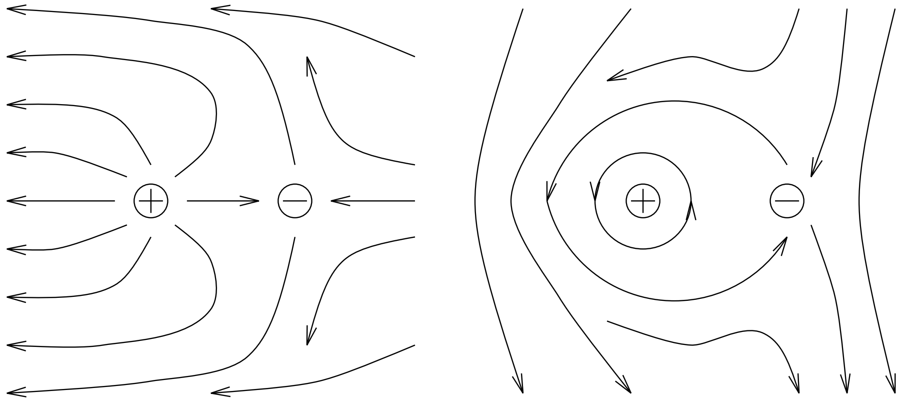

Figure 5.2: Spin configurations of vortex/antivortex pairs.

The energy cost from a single vortex of charge n has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius a to distinguish the two, i.e.

$$\boxed{\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d\mathbf{x}(\nabla\theta)^2 = \beta E_n^0(a) + \pi K n^2 \ln\left(\frac{L}{a}\right).}$$

The dominant part of the energy comes from the region outside the core and diverges logarithmically with the system size L.[3] The large energy cost associated with the defects prevents their spontaneous formation close to zero temperature. The partition function for a configuration with a single vortex of charge n is

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right], \tag{5.17}$$

where the factor of $(L/a)^2$ results from the *configurational entropy* of possible vortex locations in an area of size L^2. The entropy and energy of a vortex both grow as $\ln L$, and the free energy is dominated by one or the other. At low temperatures, large K, energy dominates and $\mathcal{Z}_1$, a measure of the weight of configurations with a single vortex, vanishes. At high enough temperatures, $K < K_n = 2/(\pi n^2)$, the entropy contribution is large enough to favour spontaneous formation of vortices. On increasing temperature, the first vortices that appear correspond to $n = \pm 1$ at $K_c = 2/\pi$. Beyond this point many vortices appear and Eq. (5.17) is no longer applicable.

In fact this estimate of K_c represents only a *lower bound* for the stability of the system towards the condensation of topological defects. This is because pairs (dipoles) of defects may appear at larger couplings. Consider a pair of charges ± 1 separated by a distance

[3] Notice that if the spin degrees of freedom have three components or more, the energy cost of a defect is finite.

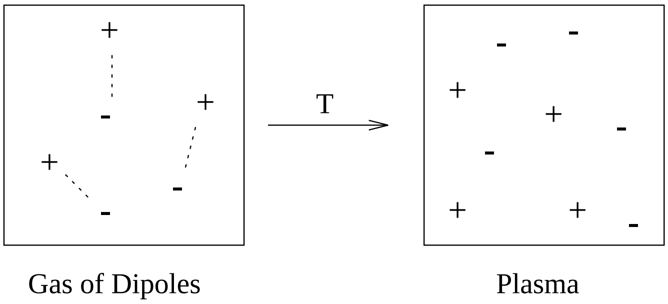

Figure 5.3: Schematic diagram showing the deconfinement of vortex pairs.

d. Distortions far from the core $|\mathbf{r}| \gg d$ can be obtained by superposing those of the individual vortices (see fig. 5.2)

$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \approx 2\mathbf{d}\cdot\nabla\left(\frac{\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z}{|\mathbf{r}|}\right),$$

which decays as $d/|\mathbf{r}|^2$. Integrating this distortion leads to a *finite* energy, and hence dipoles appear with the appropriate Boltzmann weight at any temperature. The low temperature phase should therefore be visualised as a gas of tightly bound dipoles (see fig. 5.3), their density and size increasing with temperature. The high temperature phase constitutes a plasma of unbound vortices. A theory of the Berezinskii-Kosterlitz-Thouless transition based on an RG description can be found in Chaikin and Lubensky.

5.3.1 Coulomb Gas Description of the XY Model

Vortex Interactions: In deriving long-distance vortex-vortex interactions, we can take the continuum $a \to 0$ and thermodynamic $L \to \infty$ limits. The flow field $\mathbf{v}$ of a single vortex with integer charge q at $\mathbf{r} = \mathbf{0}$ satisfies the following equation

$$\mathbf{v} = \nabla\theta = \frac{q}{r}\mathbf{e}_\phi, \tag{5.18}$$

with

$$\nabla\times\mathbf{v} = 2\pi q\delta(\mathbf{r})\mathbf{e}_z. \tag{5.19}$$

For multiple vortices with charges q_i at locations $\mathbf{r}_i$, we therefore have

$$\boxed{\nabla\times\mathbf{v} = 2\pi\sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i)\mathbf{e}_z.} \tag{5.20}$$

We now introduce a scalar potential $\Psi(\mathbf{r})$ to parametrise the flow field $\mathbf{v}$

$$\mathbf{v} = \nabla\times\Psi\mathbf{e}_z. \tag{5.21}$$

Eq. (5.20) thus leads to Laplace's equation for $\Psi(\mathbf{r})$ with the following unique solution

$$\boxed{\begin{aligned}\nabla^2\Psi &= -2\pi\sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i),\\ \Psi &= -\sum_i q_i \ln|\mathbf{r}-\mathbf{r}_i|.\end{aligned}} \tag{5.22}$$

The energy of this multiple vortex configuration is given by

$$\begin{aligned}\beta H &= \frac{K}{2}\int d^2\mathbf{r}\,\mathbf{v}^2 = \frac{K}{2}\int d^2\mathbf{r}\,\left[(\partial_x\Psi)^2+(\partial_y\Psi)^2\right]\\ &= -\frac{K}{2}\int d^2\mathbf{r}\,\Psi\left(\partial_x^2+\partial_y^2\right)\Psi + \frac{K}{2}\int_S (\Psi\nabla\Psi)\cdot d\mathbf{S}\\ &= -\pi K\sum_{i,j} q_i q_j \ln|\mathbf{r}_i-\mathbf{r}_j|, \end{aligned} \tag{5.23}$$

where the surface integral vanishes for configurations that are overall charge neutral $\sum_i q_i = 0$ (we have imposed periodic boundary conditions). As expected, the above formula gives us a divergent result for $\mathbf{r}_i = \mathbf{r}_j$ (because $L/a \to \infty$ in the continuum picture), but gives the correct asymptotic limit ($|\mathbf{r}_i - \mathbf{r}_j|/a \gg 1$) of the vortex-vortex interaction.

In the large K limit vortices will come in tightly bound vortex-antivortex pairs. We can regularise the above expression by considering the energy of a single vortex-antivortex pair

$$\beta H_{\text{pair}} = 2Eq^2 - 4\pi^2 q^2 K C(\mathbf{x}), \tag{5.24}$$

where q, $-q$ are the charges of the vortex and the anitvortex respectively, and $2E$ is the self-energy of a dipole of size a. The function $C(\mathbf{x}) = 0$ for $|\mathbf{x}| \le a$ and $C(\mathbf{x}) = \frac{1}{2\pi}\ln(|\mathbf{x}|/a)$ for $|\mathbf{x}| \ge a$. Note that we also need to enforce that the vortex separation is $|\mathbf{x}| \ge a$, because otherwise the vortices could annihilate and the above Hamiltonian would no longer give the correct energy.

In a system with multiple vortex-antivortex pairs we then have

$$\beta H = E\sum_i q_i^2 - 2\pi^2 K\sum_{i\neq j} q_i q_j C(\mathbf{x}_i - \mathbf{x}_j), \tag{5.25}$$

where E can now be interpreted as the core vortex self-energy. As the vortex-antivortex separation is increased the energy increases logarithmically indicating that there are 2D Coulomb forces between the vortices which are inversely proportional to their separation. It is important to note that the parameter K that determines the long-distance vortex interactions is the same parameter that enters the long-wavelength Gaussian limit of the original XY model, whereas the parameter E that determines the vortex core energy is directly related to the nearest-neighbour microscopic interactions of the original XY model (a cosine potential).

Dilute 2D Coulomb Gas: In the large K limit, vortices are dilute and come in tightly bound vortex-antivortex pairs. In this limit, we can write down the following partition function as a good approximation of the XY model

$$\boxed{\mathcal{Z} = \sum_{N=0}^{\infty} \frac{y^N}{((N/2)!)^2} \int \prod_{i=1}^{N} d^2\mathbf{x}_i \, e^{2\pi^2 K \sum_{i\neq j} q_i q_j C(\mathbf{x}_i - \mathbf{x}_j)},} \tag{5.26}$$

where $y = e^{-E}$ is the vortex *fugacity* and $\mathcal{Z}$ is the partition function of the dilute 2D Coulomb gas. In the limit of $y \ll 1$ configurations with $q_i = \pm 1$ dominate and only such configurations are included in the partition function. Furthermore, as explained above, we enforce charge neutrality $\sum_i q_i = 0$.

We want to explore the instability of the XY model around the fixed point $y = 0$, $K = 2/\pi$ caused by unbinding of vortex-antivortex pairs. The singular properties associated with this point are therefore captured by the above partition function, valid in the limit $y \ll 1$. Thus, the critical properties of the XY model are those of the dilute 2D Coulomb gas. We now explore this critical behaviour by following an RG scheme, originally due to Kosterlitz.

5.3.2 Perturbative RG for the Dilute Coulomb Gas

The RG scheme can be summarised as follows. We first integrate out vortex-antivortex pairs whose size ranges from a to ba and look at the renormalisation of K that results. We then rescale $\mathbf{x} = b\mathbf{x}'$ to restore the original cut-off a, which leads to the renormalisation of fugacity y.

Renormalisation of K: We will follow a slightly indirect approach here. We will introduce two external unit charges to the Coulomb gas at positions $\mathbf{x}$ and $\mathbf{x}'$ and compute their potential energy $V(\mathbf{x}, \mathbf{x}')$. This is a *physically* measurable quantity and must be constant under renormalisation. By looking at how screening from vortex-antivortex pairs contributes to this potential energy, we can *deduce* the RG transformation for K. Perturbatively in the fugacity y we only need to include corrections from a single vortex-antivortex pair

$$\begin{aligned} e^{-\beta V(\mathbf{x}-\mathbf{x}')} &= e^{-4\pi^2 K C(\mathbf{x}-\mathbf{x}')} \times \\ &\frac{\left[1 + y^2 \int d^2\mathbf{y} d^2\mathbf{y}' e^{-4\pi^2 K C(\mathbf{y}-\mathbf{y}') + 4\pi^2 K [C(\mathbf{x}-\mathbf{y}) - C(\mathbf{x}-\mathbf{y}') - C(\mathbf{x}'-\mathbf{y}) + C(\mathbf{x}'-\mathbf{y}')]} + \mathcal{O}(y^4)\right]}{\left[1 + y^2 \int d^2\mathbf{y} d^2\mathbf{y}' e^{-4\pi^2 K C(\mathbf{y}-\mathbf{y}')} + \mathcal{O}(y^4)\right]} \\ &= e^{-4\pi^2 K C(\mathbf{x}-\mathbf{x}')} \left[1 + y^2 \int d^2\mathbf{y} d^2\mathbf{y}' e^{-4\pi^2 K C(\mathbf{y}-\mathbf{y}')} \left(e^{4\pi^2 K D(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')} - 1\right) + \mathcal{O}(y^4)\right]. \end{aligned} \tag{5.27}$$

In the small fugacity limit, the size of the *internal* vortex-antivortex dipoles $\mathbf{r} = \mathbf{y}' - \mathbf{y}$ is small. We can thus approximate

$$D(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}') = C(\mathbf{x} - \mathbf{R} + \frac{\mathbf{r}}{2}) - C(\mathbf{x} - \mathbf{R} - \frac{\mathbf{r}}{2}) - C(\mathbf{x}' - \mathbf{R} + \frac{\mathbf{r}}{2}) + C(\mathbf{x}' - \mathbf{R} - \frac{\mathbf{r}}{2})$$

as:

$$-\mathbf{r}\cdot\nabla_{\mathbf{R}}C(\mathbf{x}-\mathbf{R})+\mathbf{r}\cdot\nabla_{\mathbf{R}}C(\mathbf{x}'-\mathbf{R})+\mathcal{O}(r^3) \tag{5.28}$$

where $\mathbf{R}=(\mathbf{y}+\mathbf{y}')/2$ is the dipole centre of mass. Substituting the dipole approximation for $D(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')$ into Eq. (5.27), we obtain

$$\begin{aligned} e^{-\beta V(\mathbf{x}-\mathbf{x}')} &= e^{-4\pi^2 KC(r)}\Big(1+8\pi^4K^2y^2\int d^2\mathbf{r}\, d^2\mathbf{R}\, e^{-4\pi^2KC(r)} \\ &\quad \times\left[\mathbf{r}\cdot\nabla_{\mathbf{R}}C(\mathbf{x}-\mathbf{R})-\mathbf{r}\cdot\nabla_{\mathbf{R}}C(\mathbf{x}'-\mathbf{R})\right]^2\Big), \end{aligned} \tag{5.29}$$

where the integral over the linear term in $\mathbf{r}$ vanishes. Carrying out the angular part of the $\mathbf{r}$ integration, we obtain

$$\begin{aligned} e^{-\beta V(\mathbf{x}-\mathbf{x}')} &= e^{-4\pi^2 KC(r)}\Big(1+8\pi^5K^2y^2\int r^3 e^{-4\pi^2KC(r)}dr \\ &\quad \times\int d^2\mathbf{R}\,\left[\nabla_{\mathbf{R}}C(\mathbf{x}-\mathbf{R})-\nabla_{\mathbf{R}}C(\mathbf{x}'-\mathbf{R})\right]^2. \end{aligned} \tag{5.30}$$

The second integral is proportional to the energy of a vortex and anti-vortex at locations $\mathbf{x}$ and $\mathbf{x}'$ (see Eq. (5.23)), and is equal to $2C(\mathbf{x}-\mathbf{x}')$ in the long distance limit. Note that $C(\mathbf{0})=0$.

We thus obtain the following correction to the potential $V(\mathbf{x},\mathbf{x}')$ due to screening from internal charges

$$\beta V(\mathbf{x}-\mathbf{x}')=4\pi^2C(\mathbf{x}-\mathbf{x}')\left[K-4\pi^3K^2y^2\int_a^\infty r^3\, e^{-4\pi^2KC(r)}dr\right]. \tag{5.31}$$

Hence, if dipoles ranging from a to ba are removed from the theory, K needs to be reduced by the following amount if we are to obtain the same *physical* potential from the partition function

$$\begin{aligned} \delta K &= -4\pi^3K^2y^2\int_a^{ba} r^3\, e^{-4\pi^2KC(r)}dr \\ &= -4\pi^3K^2y^2a^4\delta l, \end{aligned} \tag{5.32}$$

where $b=e^l$.

Restoring the original cutoff by $\mathbf{x}=b\mathbf{x}'$:

To complete the RG transformation and restore the cutoff on vortex separation, we simply rescale space by

$$\begin{aligned} \mathbf{x} &= b\mathbf{x}', \\ |\mathbf{x}_i-\mathbf{x}_j|>ba &\rightarrow |\mathbf{x}_i'-\mathbf{x}_j'|>a. \end{aligned} \tag{5.33}$$

In the above partition function $d^2\mathbf{x}_i = b^2 d^2\mathbf{x}'_i$ and $C(\mathbf{x}_i - \mathbf{x}_j) = C((\mathbf{x}'_i - \mathbf{x}'_j)/b) = C(\mathbf{x}'_i - \mathbf{x}'_j) + \frac{1}{2\pi}\ln b$. Both can be absorbed into the fugacity. Note that overall charge neutrality leads to

$$2\pi^2 K \sum_{i\neq j} q_i q_j \frac{1}{2\pi}\ln b = \pi K \ln b \left[\left(\sum_i q_i\right)^2 - \sum_i q_i^2\right] = -N\pi K \ln b. \tag{5.34}$$

We thus conclude that the following replacement of the partition function can be made

$$\begin{aligned}\mathcal{Z} &= \sum_{N=0}^{\infty} \frac{y^N}{((N/2)!)^2} \int \prod_{i=1}^{N} d^2\mathbf{x}_i \, e^{-2\pi^2 K \sum_{i\neq j} q_i q_j C(\mathbf{x}_i - \mathbf{x}_j)} \\ &\to \sum_{N=0}^{\infty} \frac{y'^N}{((N/2)!)^2} \int \prod_{i=1}^{N} d^2\mathbf{x}'_i \, e^{-2\pi^2 K' \sum_{i\neq j} q_i q_j C(\mathbf{x}'_i - \mathbf{x}'_j)},\end{aligned} \tag{5.35}$$

where $y' = yb^{2-\pi K}$, $K' = K - \delta K$ and the cutoff on vortex separation is a. The corresponding RG equations in the $\{K, y\}$ parameter space are

$$\boxed{\begin{aligned}\frac{dK}{dl} &= -4\pi^3 K^2 a^4 y^2 + \mathcal{O}(y^4), \\ \frac{dy}{dl} &= (2 - \pi K)y + \mathcal{O}(y^3).\end{aligned}} \tag{5.36}$$

5.3.3 Analysis of the RG Flow for the XY Model

Making the following substitution

$$\begin{aligned}K^{-1} - \pi/2 &\to x, \\ ya^2 &\to y,\end{aligned} \tag{5.37}$$

it is straightforward to show that the RG flow proceeds along hyperbolas

$$x^2 - \pi^4 y^2 = c \tag{5.38}$$

parametrised by a constant c. Fig. 5.3.3 shows the resulting RG flows in $\{x, y\}$ space.

The constant c parametrises the transition; close to the critical temperature T_c, we can write

$$c = x^2 - \pi^4 y^2 = b^2(T_c - T), \tag{5.39}$$

where $x = T - \frac{\pi}{2}$, $\ln y \propto \frac{1}{T}$ and b is a constant of the order of unity. This relation allows us to derive several of the XY model's critical properties. In particular for $c > 0$, (i.e., below the critical temperature), the RG flow terminates on the line of fixed points

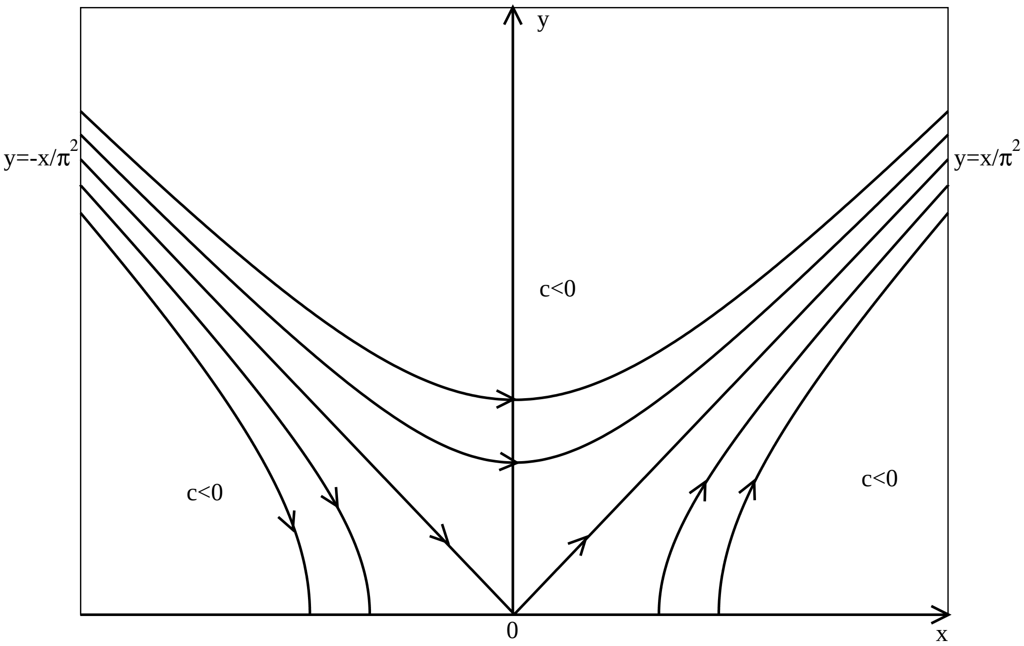

Figure 5.4: RG flow for the XY model in the Coulomb gas description limit.

formed by the negative x-axis. The effective theory below T_C is Gaussian. This is because fugacity vanishes on the line of fixed points, which means that short-distance fluctuations parametrised by E, i.e., vortex cores, in the original XY model are quenched – we are in the zero vortex sector. The effective K, which parametrises the cost of long-distance fluctuations of the original XY model is given by its fixed point K value

$$K = \frac{2}{\pi} - \frac{4}{\pi^2} \lim_{l\to\infty} x = \frac{2}{\pi} + \frac{4b}{\pi^2}\sqrt{T_c - T} \tag{5.40}$$

For $T > T_C$, K flows to zero and y flows to 1 ($E = 0$), i.e., the theory flows away from the dilute Coulomb gas limit towards the high-temperature phase with finite correlations. This is the *Debye plasma phase* where vortices, separated by distances larger than the correlation length, are completely screened from each other.

Divergence of the correlation length at $T = T_c$:

We consider an RG trajectory for $T - T_C \ll 1$, starting at $x(0) = 0$ and terminating at $x(l) = 1$, where $\xi(l) \sim a$. Substituting for y, using Eq. (5.39), into the RG flow equation for x we obtain

$$\frac{dx}{dl} = 4\pi^3 y^2 = \frac{4}{\pi}\left(x^2 + b^2(T - T_c)\right), \tag{5.41}$$

Integrating we obtain

$$\int_0^1 \frac{dx}{x^2 + b^2(T - T_c)} = \int_0^l \frac{4dl}{\pi},$$
$$\frac{1}{b^2\sqrt{T - T_C}} \arctan\left(\frac{1}{b^2\sqrt{T - T_C}}\right) = \frac{4l}{\pi}. \tag{5.42}$$

Approximating arctan $\left(\frac{1}{b^2\sqrt{T-T_C}}\right)$ by $\pi/2$ close to the critical temperature, we obtain

$$l = \frac{\pi^2}{8b^2\sqrt{T-T_C}} \tag{5.43}$$

and

$$\xi(0) = \xi(l)e^l \sim ae^{\frac{\pi^2}{8b^2\sqrt{T-T_C}}} \tag{5.44}$$

for the divergence of the correlation length at the critical point.

5.3.4 Debye Plasma Phase

For $T > T_c$, the Coulomb gas model flows to the limit where $E \to 0$. We now look more closely at the Coulomb gas Hamiltonian in this limit

$$\beta H = E\sum_i q_i^2 - 2\pi^2 K \sum_{i\neq j} q_i q_j C_2\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{a}\right). \tag{5.45}$$

For $E \to 0$ vortices proliferate and the sum over q_i in the partition function can be replaced by an integral

$$\mathcal{Z}_{\rm Db} = \int \mathcal{D}q(\mathbf{x}) \int \mathcal{D}\Psi(\mathbf{x})\, e^{\int d^2\mathbf{x}\left(-Eq(\mathbf{x})^2 + i\Psi(\mathbf{x})q(\mathbf{x})\right)} e^{-\frac{1}{4\pi^2 K}\int d^2\mathbf{x}\Psi(\mathbf{x})\nabla^2\Psi(\mathbf{x})}, \tag{5.46}$$

where a Hubbard-Stratonovich field $\Psi(\mathbf{x})$ has been introduced. Integrating out the vortex charges $q(\mathbf{x})$, we obtain

$$\mathcal{Z}_\Psi = \int \mathcal{D}\Psi(\mathbf{x})\, e^{-\int d^2\mathbf{x}\Psi(\mathbf{x})\left(\frac{1}{4E} + \frac{1}{4\pi^2 K}\nabla^2\right)\Psi(\mathbf{x})}. \tag{5.47}$$

The interaction between two static unit charges of opposite sign introduced into the system at positions $\mathbf{x}$ and $\mathbf{x}'$ is given by the following correlator

$$\begin{aligned} e^{-\beta V(\mathbf{y}-\mathbf{y}')} &= \langle e^{i\Psi(\mathbf{y}) - i\Psi(\mathbf{y}')}\rangle_{\mathcal{Z}_\Psi}, \\ \beta V(\mathbf{y}-\mathbf{y}') &= \langle \Psi(\mathbf{y})\Psi(\mathbf{y}'))\rangle_{\mathcal{Z}_\Psi} - \langle \Psi(0)^2\rangle_{\mathcal{Z}_\Psi} = e^{-|\mathbf{y}-\mathbf{y}'|/\xi} C(|\mathbf{y}-\mathbf{y}'|), \end{aligned} \tag{5.48}$$

where $\xi = \sqrt{\frac{E}{\pi^2 K}}$ is the Debye screening length and the potential energy has the form of the 2D Ornstein-Zernike correlator that we have encountered earlier.

5.4 3D Coulomb Gas

In 3D the Coulomb gas Hamiltonian takes the following form

$$\beta H = E\sum_i q_i^2 - K\sum_{i\neq j} q_i q_j \left(\frac{a}{|\mathbf{x}_i - \mathbf{x}_j|} - 1\right), \tag{5.49}$$

where position vectors $\mathbf{x}_i$ span three-dimensional space and a is the cutoff on the separation of charges. A simple application of the RG scheme used for the 2D Coulomb gas model (see Problem Set 2) shows that the there is no stable Coulomb phase. E always flow to zero towards the Debye phase fixed point. The fact that the 3D Coulomb gas is always in the Debye phase has important ramifications for 3D lattice gauge theories, which as we will see in the next chapter can be mapped onto 2D quantum electrodynamics.

3D Lattice Gauge Theory Hamiltonian: A 3D lattice gauge theory has the following Hamiltonian

$$\beta H = -K \sum_{\mathrm{P}} \cos\left(\mathrm{curl}_{\mathrm{P}}\theta\right), \tag{5.50}$$

where the variables θ live, say, on the links of a 3D simple cubic lattice, e.g., θ_{ij} lives on the link between the ith and the jth sites. The curl is taken around each face of the cubic units making up the lattice; e.g., for face P with corners at sites $i = 1, 2, 3, 4$, we have

$$\mathrm{curl}_{\mathrm{P}}\theta = \theta_1 - \theta_2 + \theta_3 - \theta_4. \tag{5.51}$$

Gaussian Limit: In the large K limit, the low energy fluctuations will be long-wavelength and, as usual, we can take the continuum Gaussian approximation

$$\begin{aligned} &\theta_{ij} = \mathbf{A}\cdot\mathbf{e}_{ij} \\ &\mathrm{curl}_{\mathrm{P}}\theta = (\nabla\times\mathbf{A})\cdot\hat{\mathbf{n}} \\ &\beta H = \frac{K}{2}\int d^3\mathbf{r}\,(\nabla\times\mathbf{A})^2 = \frac{K}{2}\int d^3\mathbf{r}\,\mathbf{B}^2\,, \end{aligned} \tag{5.52}$$

where $\mathbf{e}_{ij}$ is the vector joining lattice sites i and j, $\hat{\mathbf{n}}$ is the unit vector normal to face P and $\mathbf{B} = \nabla\times\mathbf{A}$ is the magnetic field. The connection with Maxwellian electromagnetism is now clear.

Just as before, higher order gradient terms are irrelevant and we need to look for topological defects to determine the RG flow around the $K = \infty$ fixed point.

Magnetic Monopoles: Just like $\nabla\theta$ was only measured modulo 2π for the XY model, the magnetic field $(\nabla\times\mathbf{A})$ is now only measured modulo 2π. This is because the Hamiltonian is periodic (compact gauge theory) and the energy cost of a magnetic field which is an integer multiple of 2π is zero. Let us therefore consider a pair of magnetic monopoles of charge $\pm 2\pi$. Since the magnetic field has no divergence, these have to be connected by a magnetic field line of strength 2π, known as the Dirac string. Thus for monopoles at positions $\mathbf{x}$ and $\mathbf{x}'$ the divergenceless magnetifc field configuration is given by

$$\mathbf{B}(\mathbf{r}) = \frac{(\mathbf{r}-\mathbf{x})}{2|\mathbf{r}-\mathbf{x}|^3} - \frac{(\mathbf{r}-\mathbf{x}')}{2|\mathbf{r}-\mathbf{x}'|^3} + 2\pi\Theta(z)\delta(x)\delta(y)\hat{\mathbf{e}}_z, \tag{5.53}$$

where the monopoles are separated by a distance d in the z-direction, $\mathbf{x}-\mathbf{x}' = d\hat{\mathbf{e}}_z$, and $\Theta(z) = 1$ for $\mathbf{x}'\cdot\hat{\mathbf{e}}_z < z < \mathbf{x}\cdot\hat{\mathbf{e}}_z$ and vanishes otherwise.

Because the Hamiltonian is periodic in $\mathbf{B}$, this string contributes no energy, regardless of its length (i.e., has zero tension) and can be simply neglected. The only interaction between the magnetic monopoles is now the usual 3D Coulomb attraction, obtained by integrating $\frac{K}{2}\mathbf{B}^2$ over all space.

Magnetic Monopole Gas: The defects in the lattice gauge theory form a 3D Coulomb gas. But this is always in the Debye phase! This means that magnetic monopoles proliferate and screen any externally placed static monopoles. The Gaussian expansion of the 3D lattice gauge theory is unstable and we do not obtain ordinary Maxwellian electromagnetism for any finite K. What is more, the proliferation of magnetic monopoles (large fluctuations of the magnetic field) means that in the corresponding 2D quantum electrodynamics the conjugate electric field is confined to narrow tubes (small fluctuations of the conjugate variable by Heisenberg uncertainty principle) and externally placed static electric charges feel a force that increases linearly with their separation. This is somewhat akin to quark confinement that takes place in Yang-Mills theory.