Chapter 2

Ginzburg-Landau Phenomenology

The divergence of the correlation length in the vicinity of a second-order phase transition indicates that the properties of the critical point are insensitive to microscopic details of the system. This redundancy of information motivates the search for a phenomenological description of critical phenomena which is capable of describing a wide range of model systems. In this chapter we introduce and investigate such a phenomenology known as the Ginzburg-Landau theory. Here we will explore the 'mean-field' properties of the equilibrium theory and investigate the influence of spatial fluctuations.

2.1 Ginzburg-Landau Theory

Consider the magnetic properties of a metal, say iron, close to its Curie point. The microscopic origin of magnetism is quantum mechanical, involving such ingredients as itinerant electrons, their spin, and the exclusion principle. Clearly a microscopic description is complicated, and material dependent. Such a theory would be necessary if we are to establish which elements are likely to produce ferromagnetism. However, if we accept that such behaviour exists, a microscopic theory is not necessarily the most useful way to describe its disappearance as a result of thermal fluctuations. This is because the (quantum) statistical mechanics of a collection of interacting electrons is excessively complicated. However, the degrees of freedom which describe the transition are long-wavelength collective excitations of spins. We can therefore "coarse-grain" the magnet to a scale much larger than the lattice spacing, and define a magnetisation vector field $\mathbf{m}(\mathbf{x})$, which represents the *average* of the elemental spins in the vicinity of a point $\mathbf{x}$. It is important to emphasise that, while $\mathbf{x}$ is treated as a continuous variable, the functions $\mathbf{m}$ do not exhibit any variations at distances of the order of the lattice spacing a, i.e. their Fourier transforms involve wavevectors with magnitude less than some upper cut-off $\Lambda \sim 1/a$.

In describing other types of phase transitions, the role of $\mathbf{m}(\mathbf{x})$ is played by the appropriate order parameter. For this reason it is useful to examine a generalised magnetisation vector field involving n-component magnetic moments existing in a d-dimensional space, i.e.

$$\mathbf{x} \equiv (x_1, \cdots x_d) \in \mathbb{R}^d \quad \text{(space)}, \qquad \mathbf{m} \equiv (m_1, \cdots m_n) \in \mathbb{R}^n \quad \text{(spin)}.$$

Some specific problems covered in this framework include:

$n = 1$: Liquid-gas transitions; binary mixtures; and uniaxial magnets;

$n = 2$: Superfluidity; superconductivity; and planar magnets;

$n = 3$: Classical isotropic magnets.

While most applications occur in three-dimensions, there are also important phenomena on surfaces ($d = 2$), and in wires ($d = 1$). (Relativistic field theory is described by a similar structure, but in $d = 4$.)

A general coarse-grained Hamiltonian can be constructed on the basis of appropriate symmetries:

1. **Locality**: The Hamiltonian should depend on the local magnetisation and short range interactions expressed through gradient expansions:

$$\beta H = \int d\mathbf{x}\; f[\mathbf{m}(\mathbf{x}), \nabla \mathbf{m}, \cdots]$$

2. **Rotational Symmetry**: Without magnetic field, the Hamiltonian should be isotropic in space and therefore invariant under rotations, $\mathbf{m} \mapsto \mathbf{R}_n \mathbf{m}$.

$$\beta H[\mathbf{m}] = \beta H[\mathbf{R}_n \mathbf{m}].$$

3. **Translational and Rotational Symmetry in $\mathbf{x}$**: This last constraint finally leads to a Hamiltonian of the form

$$\beta H = \int d\mathbf{x}\; \Big[\tfrac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + \cdots + \tfrac{K}{2}(\nabla \mathbf{m})^2 + \tfrac{L}{2}(\nabla^2 \mathbf{m})^2 + \tfrac{N}{2}\mathbf{m}^2(\nabla \mathbf{m})^2 + \cdots - \mathbf{h}\cdot\mathbf{m}\Big]. \quad (2.1)$$

(Recall that, as we are primarily interested in transitions between different phases and the fields are the order parameters of these transitions, we can work close enough to the critical point to be able to assume that the magnitude of the fields and of their derivatatives are small and can be expanded upon.) Equation (2.1) is known as the **Ginzburg-Landau Hamiltonian**. It depends on a set of *phenomenological* parameters t, u, K, etc. which are *non-universal* functions of microscopic interactions, *as well as external parameters such as temperature, and pressure.*[1]

[1]It is essential to appreciate the latter point, which is usually the source of much confusion. The probability for a particular configuration of the field is given by the Boltzmann weight $\exp\{-\beta H[\mathbf{m}(\mathbf{x})]\}$. This does NOT imply that all terms in the exponent are proportional to $(k_B T)^{-1}$. Such dependence

2.2 Landau Mean-Field Theory

The original problem has been simplified considerably by focusing on the coarse-grained magnetisation field described by the Ginzburg-Landau Hamiltonian. The various thermodynamic functions (and their singular behaviour) can now be obtained from the corresponding partition function

$$\boxed{\mathcal{Z}[T, \mathbf{h}] = \int D\mathbf{m}(\mathbf{x})\ e^{-\beta H[\mathbf{m},\mathbf{h}]}} \tag{2.2}$$

Since the degrees of freedom appearing in the Hamiltonian are functions of $\mathbf{x}$, the symbol $\int D\mathbf{m}(\mathbf{x})$ refers to the **functional integral**. As such, it should be regarded as a limit of discrete integrals, i.e., for a one-dimensional Hamiltonian,

$$\int D\mathbf{m}(\mathbf{x})\ z[\mathbf{m}(\mathbf{x}), \partial\mathbf{m}, \cdots] \equiv \lim_{a\to 0, N\to\infty} \int \prod_{i=1}^{N} d\mathbf{m}_i\ z[\mathbf{m}_i, (\mathbf{m}_{i+1} - \mathbf{m}_i)/a, \cdots].$$

In general, evaluating the functional integral is not straightforward. However, we can obtain an estimate of $\mathcal{Z}$ by applying a saddle-point or mean-field approximation to the functional integral

$$\boxed{\mathcal{Z}[T, \mathbf{h}] \equiv e^{-\beta F[T,\mathbf{h}]}, \qquad \beta F[T, \mathbf{h}] \simeq \min_{\mathbf{m}} \left[\beta H\left[\mathbf{m}, \mathbf{h}\right]\right],}$$

where $\min_{\mathbf{m}}[\beta H[\mathbf{m}, \mathbf{h}]]$ represents the minimum of the function with respect to variations in $\mathbf{m}$. Such an approach is known as **Landau mean-field theory**. For $K > 0$, the minimum free energy occurs for a uniform vector field $\mathbf{m}(\mathbf{x}) \equiv \bar{m}\hat{\mathbf{e}}_h$, where $\hat{\mathbf{e}}_h$ points along the direction of the magnetic field, and $\bar{m}$ is obtained by minimizing the **Landau free energy density**

$$f(m, h) \equiv \frac{\beta F}{V} = \tfrac{t}{2}m^2 + um^4 - hm$$

In the vicinity of the critical point $\bar{m}$ is small, and we are justified in keeping only the lowest powers in the expansion of $f(m, h)$. The behaviour of $f(m, h)$ depends sensitively on the sign of t (see Fig. 2.1).

1. For $t > 0$ the quartic term can be ignored, and the minimum occurs for $\bar{m} \simeq h/t$. The vanishing of the magnetisation as $h \to 0$ signals paramagnetic behaviour, and the zero-field susceptibility $\chi \sim 1/t$ diverges as $t \to 0$.

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holds only for the true microscopic Hamiltonian. The Ginzburg-Landau Hamiltonian is more accurately regarded as an effective free energy obtained by integrating over the microscopic degrees of freedom (coarse-graining), while constraining their average to $\mathbf{m}(\mathbf{x})$. It is precisely because of the difficultly of carrying out such a first principles program that we postulate the form of the resulting effective free energy on the basis of symmetries alone. The price we pay is that the phenomenological parameters have an unknown functional dependence on the original microscopic parameters, as well as on external constraints such as temperature (since we have to account for the entropy of the short distance fluctuations lost in the coarse-graining procedure).

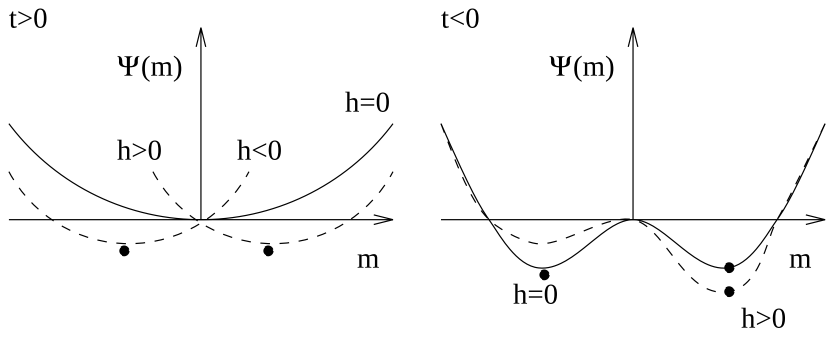

Figure 2.1: Schematic diagram of the mean-field Landau free energy.

2. For $t < 0$ a quartic term with a positive value of u is required to ensure stability (i.e. to keep the magnetisation finite). The function $f(m, h)$ now has degenerate minima at a non-zero value of $\bar{m}$. At $h = 0$ there is a *spontaneous breaking of rotational symmetry in spin space* indicating ordered or ferromagnetic behaviour. Switching on an infinitesimal field h leads to a realignment of the magnetisation along the field direction and breaks the degeneracy of the ground state.

Thus a saddle-point evaluation of the Ginzburg-Landau Hamiltonian suggests paramagnetic behaviour for $t > 0$, and ferromagnetic behaviour for $t < 0$. Without loss of generality (i.e. by adjusting the scale of the order parameter), we can identify the parameter t with the reduced temperature $t = (T - T_c)/T_c$. More generally, we can map the phase diagram of the Ginzburg-Landau Hamiltonian to that of a magnet by setting

$$\begin{aligned} t(T, \cdots) &= (T - T_c)/T_c + O(T - T_c)^2, \\ u(T, \cdots) &= u_0 + u_1(T - T_c) + O(T - T_c)^2, \\ K(T, \cdots) &= K_0 + K_1(T - T_c) + O(T - T_c)^2, \end{aligned}$$

where u_0, K_0 are unknown positive constants depending on material properties of the system. With this identification, let us determine some of the thermodynamical properties implied by the mean-field analysis.

▷ **Magnetisation**: An explicit expression for the average magnetisation $\bar{m}$ can be found from the stationary condition

$$\left.\frac{\partial f}{\partial m}\right|_{m=\bar{m}} = 0 = t\bar{m} + 4u\bar{m}^3 - h.$$

In zero magnetic field we find

$$\boxed{\bar{m} = \begin{cases} 0 & t > 0, \\ \sqrt{-t/4u} & t < 0, \end{cases}}$$

which implies a universal exponent of $\beta = 1/2$, while the amplitude is material dependent.

▷ **Heat Capacity**: For $h = 0$, the free energy density is given by

$$f(m, h=0) \equiv \frac{\beta F}{V}\Big|_{h=0} = \begin{cases} 0 & t > 0, \\ -t^2/16u & t < 0. \end{cases}$$

Thus, by making use of the identities

$$E = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}, \qquad \frac{\partial}{\partial \beta} = -k_B T^2 \frac{\partial}{\partial T} \simeq -k_B T_c \frac{\partial}{\partial t},$$

the singular contribution to the heat capacity is found to be

$$C_{\text{sing.}} = \frac{\partial E}{\partial T} = \begin{cases} 0, & t > 0, \\ k_B/8u & t < 0. \end{cases}$$

This implies that the specific heat exponents $\alpha_+ = \alpha_- = 0$. In this case we observe only a discontinuity in the singular part of the specific heat. However, notice that by including higher order terms, we can in principle obtain non-zero critical exponents.

▷ **Susceptibility**: The magnetic response is characterised by the (longitudinal) susceptibility

$$\chi_l \equiv \frac{\partial \bar{m}}{\partial h}\Big|_{h=0}, \qquad \chi_l^{-1} = \frac{\partial h}{\partial \bar{m}}\Big|_{h=0} = t + 12u\bar{m}^2\Big|_{h=0} = \begin{cases} t & t > 0, \\ -2t & t < 0, \end{cases}$$

which, as a measure of the variance of the magnetisation, must be positive. From this expression, we can deduce the critical exponents $\gamma_+ = \gamma_- = 1$. Although the amplitudes are parameter dependent, their ratio $\chi_l^+/\chi_l^- = 2$ is also universal.

▷ **Equation of State**: Finally, on the critical isotherm, $t = 0$, the magnetisation behaves as

$$\bar{m} = \left(\frac{h}{4u}\right)^{1/3} \sim h^{1/\delta}.$$

giving the exponent $\delta = 3$.

This completes our survey of the critical properties of the Ginzburg-Landau theory in the Landau mean-field approximation. To cement these ideas one should attempt to find the mean-field critical exponents associated with a **tricritical point** (see, for example, the first problem set).To complement these notes it is also useful to refer to Section 4.2 (p151–154) of Chaikin and Lubensky on Landau theory.

Landau mean-field theory accommodates only the minimum energy configuration. To test the validity of this approximation scheme, and to determine spatial correlations it is necessary to take into account configurations of the field $\mathbf{m}(\mathbf{x})$ involving spatial fluctuations. However, before doing so, it is first necessary to acquire some familiarity with the method of Gaussian functional integration, the basic machinery of statistical (and quantum) field theory.

2.3 Gaussian and Functional Integration

▷ INFO: Before defining the Gaussian functional integral, it is useful to recall some results involving integration over discrete variables. We begin with the Gaussian integral involving a single (real) variable ϕ,

$$\mathcal{Z}_1 = \int_{-\infty}^{\infty} d\phi \ \exp\left[-\frac{\phi^2}{2G} + h\phi\right] = \sqrt{2\pi G} \ \exp\left[\frac{Gh^2}{2}\right].$$

Now derivatives of $\mathcal{Z}_1$ on h generate Gaussian integrals involving powers of ϕ. Thus, if the integrand represents the probability distribution of a random variable ϕ, logarithmic derivatives can be used to generate moments ϕ. In particular,

$$\langle \phi \rangle \equiv \frac{\partial \ln \mathcal{Z}_1}{\partial h} = hG.$$

Subjecting $\ln \mathcal{Z}_1$ to a second derivative obtains (exercise)

$$\frac{\partial^2 \ln \mathcal{Z}_1}{\partial h^2} = \langle \phi^2 \rangle - \langle \phi \rangle^2 = G.$$

Note that, in general, the second derivative does not simply yield the second moment. In fact it obtains an object known as the 'second cumulant', the physical significance of which will become clear later. However, in the present case, it is simple to deduce from the expansion, $\langle \phi \rangle = hG$, and $\langle \phi^2 \rangle = h^2G^2 + G$.

Higher moments are more conveniently expressed by the **cumulant expansion**[2]

$$\boxed{\langle \phi^r \rangle_c = \frac{\partial^r}{\partial k^r}\Big|_{k=0} \ln \left\langle e^{k\phi} \right\rangle}$$

Applied to the first two cumulants, one obtains $\langle \phi \rangle_c = \langle \phi \rangle = hG$, and $\langle \phi^2 \rangle_c = \langle \phi^2 \rangle - \langle \phi \rangle^2 = G$ (as above), while $\langle \phi^r \rangle_c = 0$ for $r > 2$. The average $\langle e^{k\phi} \rangle$ is known as the **moment generating function**.

Gaussian integrals involving N (real) variables

$$\mathcal{Z}_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\frac{1}{2}\phi^T \mathbf{G}^{-1} \phi + \mathbf{h} \cdot \phi\right], \tag{2.3}$$

can be reduced to a product of N one-dimensional integrals by diagonalising the (real symmetric) matrix $\mathbf{G}^{-1}$. (Convergence of the Gaussian integration is assured only when the eigenvalues are positive definite.) Denoting the unitary matrices that diagonalise $\mathbf{G}$ by $\mathbf{U}$, the matrix

[2]The moments are related to the cumulants by the identity

$$\langle \phi^n \rangle = \sum_P \prod_\alpha \langle \phi^{n_\alpha} \rangle_c,$$

where $\sum_P$ represents the sum over all partitions of the product ϕ^n into subsets ϕ^{n_α} labelled by α.

$\widetilde{\mathbf{G}}^{-1} = \mathbf{U}\mathbf{G}^{-1}\mathbf{U}^{-1}$ represents the diagonal matrix of eigenvalues. Making use of the identity (i.e. completing the square)

$$\frac{1}{2}\phi^T\mathbf{G}^{-1}\phi - \mathbf{h}\cdot\phi = \chi^T\widetilde{\mathbf{G}}^{-1}\chi - \frac{1}{2}\mathbf{h}^T\mathbf{U}^{-1}\widetilde{\mathbf{G}}\mathbf{U}\mathbf{h},$$

where $\chi = \mathbf{U}\phi - \widetilde{\mathbf{G}}\mathbf{U}\mathbf{h}$, and changing integration variables (since the transformation is unitary, the corresponding Jacobian is unity) we obtain

$$\begin{aligned}\mathcal{Z}_N &= \int_{-\infty}^{\infty}\prod_{i=1}^{N} d\chi_i \exp\left[-\frac{1}{2}\chi^T\widetilde{\mathbf{G}}^{-1}\chi + \frac{1}{2}\mathbf{h}^T\mathbf{U}^{-1}\widetilde{\mathbf{G}}\mathbf{U}\mathbf{h}\right],\\ &= \det(2\pi\mathbf{G})^{1/2}\exp\left[\frac{1}{2}\mathbf{h}^T\mathbf{G}\mathbf{h}\right].\end{aligned}\tag{2.4}$$

Regarding $\mathcal{Z}_N$ as the partition function of a set of N Gaussian distributed random variables, $\{\phi_i\}$, the corresponding cumulant expansion is generated by

$$\boxed{\langle\phi_i\cdots\phi_j\rangle_c = \frac{\partial}{\partial k_i}\cdots\frac{\partial}{\partial k_j}\Big|_{\mathbf{k}=0}\ln\left\langle e^{\mathbf{k}\cdot\phi}\right\rangle,}$$

where the moment generatingfunction is equal to

$$\boxed{\left\langle e^{\mathbf{k}\cdot\phi}\right\rangle = \exp\left[\mathbf{h}^T\mathbf{G}\mathbf{k} + \frac{1}{2}\mathbf{k}^T\mathbf{G}\mathbf{k}\right].}\tag{2.5}$$

Applying this result we find that the first two cumulants are given by

$$\langle\phi_i\rangle_c = \sum_j G_{ij}h_j, \qquad \langle\phi_i\phi_j\rangle_c = G_{ij},\tag{2.6}$$

while, as for the case $N = 1$, cumulants higher than the second vanish. The latter is a unique property of Gaussian distributions. Applying Eq. (2.5), we can further deduce the important result that for any linear combination of Gaussian distributed variables $A = \mathbf{a}\cdot\phi$,

$$\boxed{\langle e^A\rangle = e^{\langle A\rangle_c + \langle A^2\rangle_c/2}.}$$

Now Gaussian **functional integrals** are a limiting case of the above. Consider the points i as the sites of a d-dimensional lattice and let the spacing go to zero. In the continuum limit, the set $\{\phi_i\}$ translates to a function $\phi(\mathbf{x})$, and the matrix G^{-1}_{ij} is replaced by an **operator** kernel or **propagator** $G^{-1}(\mathbf{x},\mathbf{x}')$. The natural generalisation of Eq. (2.4) is

$$\begin{aligned}&\int D\phi(\mathbf{x})\exp\left[-\frac{1}{2}\int d\mathbf{x}\int d\mathbf{x}'\ \phi(\mathbf{x})\ G^{-1}(\mathbf{x},\mathbf{x}')\ \phi(\mathbf{x}') + \int d\mathbf{x}\ h(\mathbf{x})\phi(\mathbf{x})\right]\\ &\qquad\qquad \propto (\det\hat{\mathbf{G}})^{1/2}\exp\left[\frac{1}{2}\int d\mathbf{x}\int d\mathbf{x}'\ h(\mathbf{x})\ G(\mathbf{x},\mathbf{x}')\ h(\mathbf{x}')\right],\end{aligned}$$

where the inverse kernel $G(\mathbf{x},\mathbf{x}')$ satisfies the equation

$$\boxed{\int d\mathbf{x}'\ G^{-1}(\mathbf{x},\mathbf{x}')\ G(\mathbf{x}',\mathbf{x}'') = \delta^d(\mathbf{x}-\mathbf{x}'').}\tag{2.7}$$

The notation $D\phi(\mathbf{x})$ is used to denote the measure of the functional integral. Although the constant of proportionality $(2\pi)^N$ left out is formally divergent in the thermodynamic limit, it does not affect averages that are obtained from derivatives of such integrals. For Gaussian distributed functions, Eq. (2.6) then generalises to

$$\boxed{\langle\phi(\mathbf{x})\rangle_c = \int d\mathbf{x}\ G(\mathbf{x},\mathbf{x}')\ h(\mathbf{x}'), \qquad \langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_c = G(\mathbf{x},\mathbf{x}').}$$

Later, in dealing with small fluctuations in the Ginzburg-Landau Hamiltonian, we will frequently encounter the quadratic form,

$$\boxed{\beta H[\phi] = \frac{1}{2}\int d\mathbf{x}\left[(\nabla\phi)^2 + \xi^{-2}\phi^2\right] \equiv \frac{1}{2}\int d\mathbf{x}\int d\mathbf{x}'\ \phi(\mathbf{x}')\delta^d(\mathbf{x}-\mathbf{x}')(-\nabla^2+\xi^{-2})\ \phi(\mathbf{x}),} \quad (2.8)$$

which (integrating by parts) implies an operator kernel

$$\boxed{G^{-1}(\mathbf{x},\mathbf{x}') = K\delta^d(\mathbf{x}-\mathbf{x}')(-\nabla^2+\xi^{-2}).}$$

Substituting into Eq. (2.7) and integrating we obtain $K(-\nabla^2+\xi^{-2})G(\mathbf{x}) = \delta^d(\mathbf{x})$. The propagator can thus be identified as nothing but the **Green's function**.

In the present case, translational invariance of the propagator suggests the utility of the Fourier representation[3]

$$\boxed{\phi(\mathbf{x}) = \sum_{\mathbf{q}}\phi_{\mathbf{q}}\ e^{i\mathbf{q}\cdot\mathbf{x}}, \qquad \phi_{\mathbf{q}} = \frac{1}{L^d}\int_0^L d\mathbf{x}\ \phi(\mathbf{x})\ e^{-i\mathbf{q}\cdot\mathbf{x}},}$$

where $\mathbf{q} = (q_1, \cdots q_d)$, with the Fourier elements taking values $q_i = 2\pi m/L$, m integer. In this representation, making use of the identity $\int_0^L d\mathbf{x}\ e^{-i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} = L^d\delta_{\mathbf{q},-\mathbf{q}'}$, the quadratic form above becomes diagonal in $\mathbf{q}$[4]

$$\beta H[\phi] = \frac{1}{2}\sum_{\mathbf{q}}(\mathbf{q}^2+\xi^{-2})|\phi_{\mathbf{q}}|^2,$$

[3]Here the system is supposed to be confined to a square box of dimension d and volume L^d. In the thermodynamic limit $L\to\infty$, the Fourier series becomes the transform

$$\boxed{\phi(\mathbf{x}) = \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^d}\ \phi(\mathbf{q})\ e^{i\mathbf{q}\cdot\mathbf{x}}, \qquad \phi(\mathbf{q}) = \int_{-\infty}^{\infty} d\mathbf{x}\ \phi(\mathbf{x})\ e^{-i\mathbf{q}\cdot\mathbf{x}}.}$$

Similarly,

$$\boxed{\int_{-\infty}^{\infty} d\mathbf{x}\ e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} = (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}'), \qquad \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^d}\ e^{-i\mathbf{q}\cdot(\mathbf{x}+\mathbf{x}')} = \delta^d(\mathbf{x}+\mathbf{x}').}$$

In the formulae above, the arrangements of $(2\pi)^d$ is not occasional. In defining the Fourier transform, it is wise to declare a convention and stick to it. The convention chosen here is one in which factors of $(2\pi)^d$ are attached to the $\mathbf{q}$ integration, and to the δ-function in $\mathbf{q}$.

[4]Similarly, in the thermodynamic limit, the Hamiltonian takes the form

$$\beta H[\phi] = \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^d}\frac{1}{2}(\mathbf{q}^2+\xi^{-2})|\phi(\mathbf{q})|^2.$$

where, since $\phi(\mathbf{x})$ is real, $\phi_{-\mathbf{q}} = \phi^*_{\mathbf{q}}$. The corresponding propagator is given by $G(\mathbf{q}) = (\mathbf{q}^2 + \xi^{-2})^{-1}$. Thus in real space, the correlation function is given by

$$G(\mathbf{x}, \mathbf{x}') \equiv \langle \phi(\mathbf{x})\phi(\mathbf{x}')\rangle_c = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} G(\mathbf{q}).$$

Here we have kept L finite and the modes discrete to emphasize the connection between the discrete Gaussian integrations $\mathcal{Z}_N$ and the functional integral. Hereafter, we will focus on the thermodynamic limit $L \to \infty$.

2.4 Derivation of GL Hamiltonian for the Ising model

We now illustrate how a coarse-grained GL Hamiltonian can arise from a microscopic model with a first principles derivation. We will use the Ising model as a convenient example. The partition function of the Ising model takes the following form

$$\mathcal{Z} = \sum_{\{\sigma_i = \pm 1\}} e^{K\sum_{<ij>} \sigma_i\sigma_j + h\sum_i \sigma_i}. \tag{2.9}$$

Step 1 — Introducing the order parameter via Hubbard-Stratonovich decoupling

$$\mathcal{Z} = \det\left[2\pi G^{-1}_{ij}\right]^{-\frac{1}{2}} \sum_{\{\sigma_i=\pm 1\}} \int \left(\prod_i d\Psi_i\right) e^{-\frac{1}{2}\sum_{ij} G^{-1}_{ij}\Psi_i\Psi_j} e^{\sum_i (\Psi_i + h)\sigma_i}, \tag{2.10}$$

where $G_{ij} = K$ for nearest neighbours ij and vanishes otherwise.

Step 2 — Integrating out the original microscopic variables

The partition function factorises, allowing us to carry out the sum over σ_i

$$\begin{aligned}\mathcal{Z} &\propto \sum_{\{\sigma_i=\pm 1\}} \int \left(\prod_i d\Psi_i\right) e^{-\frac{1}{2}\sum_{ij} G^{-1}_{ij}\Psi_i\Psi_j} \left(\prod_i e^{\sigma_i(\Psi_i+h)}\right) \\ &= \int \left(\prod_i d\Psi_i\right) e^{-\frac{1}{2}\sum_{ij} G^{-1}_{ij}\Psi_i\Psi_j} \left(\prod_i 2\cosh\left(\Psi_i + h\right)\right).\end{aligned} \tag{2.11}$$

Step 3 — Re-expontiating to obtain the GL Hamiltonian

$$\mathcal{Z} \propto \int \left(\prod_i d\Psi_i\right) e^{-\frac{1}{2}\sum_{ij} G^{-1}_{ij}\Psi_i\Psi_j} e^{\sum_i \ln\cosh(\Psi_i+h)}. \tag{2.12}$$

Step 4 — Expanding in the order parameter and its gradients close to the critical point

$$\mathcal{Z} = \int \left(\prod_i d\Psi_i\right) e^{-\frac{1}{2}\sum_{ij} G^{-1}_{ij}\Psi_i\Psi_j} e^{\sum_i\left(\frac{1}{2}\Psi_i^2 - \frac{1}{12}\Psi_i^4 + h\Psi_i\right)}, \tag{2.13}$$

where we have kept terms up to fourth order in Ψ_i (and up to linear order in h). It is convenient to work in Fourier space and to find the Fourier transform of G_{ij} before inverting it; this can be done either directly or by computing

$$\sum_{ij} G_{ij}\phi_i\phi_j = \sum_{\mathbf{q}\in\mathrm{BZ},\alpha} 2K\cos(q_\alpha a)\phi_{\mathbf{q}}\phi_{-\mathbf{q}}, \tag{2.14}$$

where $\phi_i = \frac{1}{\sqrt{N}}\sum_{\mathbf{q}\in\mathrm{BZ}} e^{i\mathbf{q}\cdot\mathbf{r}}\phi_{\mathbf{q}}$ is a generic (vector) variable and q_α is one of the d Cartesian components of $\mathbf{q}$. Sufficiently close to the critical point, the relevant modes will be long-wavelength $|\mathbf{q}|a \ll 1$ modes

$$\sum_{ij} G_{ij}\phi_i\phi_j = \sum_{\mathbf{q}\in\mathrm{BZ}} 2Kd\left(1 - \frac{|\mathbf{q}|^2a^2}{2d}\right)\Psi_{\mathbf{q}}\Psi_{-\mathbf{q}} + \mathcal{O}(|\mathbf{q}|^4a^4)\,. \tag{2.15}$$

We can now easily invert the matrix in Fourier space

$$\begin{aligned}
G_{\mathbf{q},\mathbf{q}'} &= \delta_{\mathbf{q}+\mathbf{q}'}2Kd\left(1 - \frac{|\mathbf{q}|^2a^2}{2d}\right),\\
G^{-1}_{\mathbf{q},\mathbf{q}'} &= \delta_{\mathbf{q}+\mathbf{q}'}\frac{1}{2Kd}\left(1 + \frac{|\mathbf{q}|^2a^2}{2d}\right),\\
\frac{1}{2}\sum_{ij} G^{-1}_{ij}\Psi_i\Psi_j &= \frac{1}{4Kd}\sum_{\mathbf{q}\in\mathrm{BZ}}\left(1 + \frac{|\mathbf{q}|^2a^2}{2d}\right)|\Psi_{\mathbf{q}}|^2.
\end{aligned} \tag{2.16}$$

Returning to real space and replacing sums with integrals $\sum_i \to \int \frac{d^d\mathbf{r}}{a^d}$, $\sum_{\mathbf{q}\in\mathrm{BZ}} \to \int_{\mathrm{BZ}} \frac{Na^d d^d\mathbf{q}}{(2\pi)^d}$, $\Psi_{\mathbf{q}} \to \frac{\Psi(\mathbf{q})}{\sqrt{N}a^d}$, and $\Psi(\mathbf{q}) = \int d^d\mathbf{r}\,\Psi(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}$, which is valid for long-wavelength fluctuations, we obtain

$$\mathcal{Z} \propto \int \mathcal{D}\Psi(\mathbf{r})\, e^{-\beta H[\Psi(\mathbf{r})]}, \tag{2.17}$$

where $\beta H[\Psi(\mathbf{r})]$ is the Ising model Ginzburg-Landau Hamiltonian

$$\beta H[\Psi(\mathbf{r})] = \int \frac{d^d\mathbf{r}}{a^d}\left[\frac{t}{2}\Psi^2(\mathbf{r}) + \frac{a^2}{8Kd^2}\left(\nabla\Psi(\mathbf{r})\right)^2 + \frac{1}{12}\Psi^4(\mathbf{r}) - h\Psi(\mathbf{r})\right].$$

The reduced temperature $t = \frac{1}{2Kd} - 1$ identifies $1/K_C = 2d$ as the critical temperature in the mean-field approximation.

2.5 Symmetry Breaking: Goldstone Modes

With these important mathematical preliminaries, we return to the consideration of the influence of spatial fluctuations on the stability of the mean-field analysis. Even for $h = 0$, when βH has full rotational symmetry, the ground state of the Ginzburg-Landau Hamiltonian is ordered along some given direction for $T < T_c$ — a direction of 'magnetisation' is specified. One can say that the onset of **long-range order** is accompanied by the **spontaneous breaking** of the rotational symmetry. The presence of a degenerate manifold of ground states obtained by a *global* rotation of the order parameter implies the existence of low energy excitations corresponding to slowly varying rotations in the spin space. Such excitations are characteristic of systems with a broken *continuous* symmetry and are known as **Goldstone modes**. In magnetic systems the Goldstone modes are known as **spin-waves**, while in solids, they are the vibrational or *phonon* modes.

The influence of Goldstone modes can be explored by treating fluctuations within the framework of the Ginzburg-Landau theory. For a fixed magnitude of the n-component order parameter or, in the spin model, the magnetic moment $\mathbf{m} = \bar{m}\hat{\mathbf{e}}_h$, the transverse fluctuations can be parametrized in terms of a set of $n-1$ angles. One-component, or **Ising** spins have only a discrete symmetry and possess no low energy excitations. Two-component, or XY-spins, where the moment lies in a plane, are defined by a single angle θ, $\mathbf{m} = \bar{m}(\cos\theta, \sin\theta)$ (cf. the complex phase of a 'superfluid' order parameter). In this case the Ginzburg-Landau free energy functional takes the form

$$\boxed{\beta H[\theta(\mathbf{x})] = \beta H_0 + \frac{\bar{K}}{2}\int d\mathbf{x}\ (\nabla\theta)^2,} \tag{2.18}$$

where $\bar{K} = K\bar{m}^2/2$.

Although superficially quadratic, the multi-valued nature of the transverse field $\theta(\mathbf{x})$ makes the evaluation of the partition function problematic. However, at low temperatures, taking the fluctuations of the fields to be small $\theta(\mathbf{x}) \ll 2\pi$, the functional integral can be taken as Gaussian. Following on from our discussion of the Gaussian functional integral, the operator kernel or propagator can be identified simply as the Laplacian operator. The latter is diagonalised in Fourier space, and the corresponding degrees of freedom are associated with spin-wave modes.

Then, employing the results of the previous section, we immediately find the average phase vanishes $\langle\theta(\mathbf{x})\rangle = 0$, and the correlation function takes the form

$$G(\mathbf{x}, \mathbf{x}') \equiv \langle\theta(\mathbf{x})\theta(\mathbf{x}')\rangle = -\frac{C_d(\mathbf{x}-\mathbf{x}')}{\bar{K}}, \qquad \nabla^2 C_d(\mathbf{x}) = \delta^d(\mathbf{x})$$

where C_d denotes the Coulomb potential for a δ-function charge distribution. Exploiting the symmetry of the field, and employing Gauss' law, $\int_V d\mathbf{x}\ \nabla^2 C_d(\mathbf{x}) = \oint dS\cdot\nabla C_d$, we find that C_d depends only on the radial coordinate x, and

$$\frac{dC_d}{dx} = \frac{1}{x^{d-1}S_d}, \qquad C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + \text{const.}, \tag{2.19}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of a unit d-dimensional ball.[5] Hence

$$\left\langle [\theta(\mathbf{x}) - \theta(0)]^2 \right\rangle = 2\left[\langle \theta(0)^2\rangle - \langle \theta(\mathbf{x})\theta(0)\rangle\right] \overset{|\mathbf{x}|>a}{=} \frac{2(|\mathbf{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d},$$

where the cut-off, a, is of the order of the lattice spacing. Note that, in the case where $d = 2$, the combination $|\mathbf{x}|^{2-d}/(2-d)$ must be interpreted as $\ln|\mathbf{x}|$.

The long distance behaviour changes dramatically at $d = 2$. For $d > 2$, the phase fluctuations approach some finite constant as $|\mathbf{x}| \to \infty$, while they become asymptotically large for $d \leq 2$. Since the phase is bounded by 2π, it implies that long-range order (predicted by the mean-field theory) is destroyed. This result becomes more apparent by examining the effect of phase fluctuations on the two-point correlation function,

$$\langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0)\rangle = \bar{m}^2 \mathrm{Re}\ \left\langle e^{i[\theta(\mathbf{x})-\theta(0)]}\right\rangle .$$

(Since amplitude fluctuations are neglected, we are in fact looking at the **transverse correlation function**.) For Gaussian distributed variables (with zero mean) we have already seen that $\langle \exp[\alpha\theta]\rangle = \exp[\alpha^2\langle\theta^2\rangle/2]$. We thus obtain

$$\langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0)\rangle = \bar{m}^2 \exp\left[-\frac{1}{2}\langle[\theta(\mathbf{x}) - \theta(0)]^2\rangle\right] = \bar{m}^2 \exp\left[-\frac{(x^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d}\right],$$

implying a power-law decay of correlations in $d = 2$, and an exponential decay in $d < 2$,

$$\boxed{\lim_{|\mathbf{x}|\to\infty} \langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0)\rangle = \begin{cases} \bar{m}'^2 & d > 2, \\ 0 & d \leq 2. \end{cases}}$$

The saddle-point approximation to the order parameter, $\bar{m}$ was obtained by neglecting fluctuations. The result above demonstrates that the inclusion of phase fluctuations leads to a reduction in the degree of order in $d = 2$, and to its complete destruction in $d < 2$. This result typifies a more general result known as the **Mermin-Wagner Theorem** (N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1967)). The theorem states that there is no spontaneous breaking of a continuous symmetry in systems with short-range interactions in dimensions $d \leq 2$. Corollaries to the theorem include:

[5] An important consequence of Eq. (2.19) is the existence of an unphysical **ultraviolet divergence** of the theory (i.e. $x \to 0 \longleftrightarrow q \to \infty$) in dimensions $d \geq 2$. In the present case, this divergence can be traced to the limited form of the effective free energy which accommodates short-range fluctuations of arbitrary magnitude. In principle one can account for the divergence by introducing additional terms in the free energy which control the short-range behaviour more precisely. Alternatively, and in keeping with the philosophy that lies behind the Ginzburg-Landau theory, we can introduce a short-length scale cut-off into the theory, a natural candidate being the "lattice spacing" a of the coarse-grained free energy. Note, however, that were the free energy a microscopic one — i.e. a free field theory — we would be forced to make sense of the ultraviolet divergence. Indeed finding a renormalisation scheme to control ultraviolet aspects of the theory is the subject of high energy quantum field theory. In condensed matter physics our concern is more naturally with the **infrared**, long-wavelength divergence of the theory which, in the present case (2.19), appears in dimensions $d \leq 2$.

- The borderline dimensionality of two, known as the **lower critical dimension** d_l has to be treated carefully. As we shall show in Chapter 5, there is in fact a phase transition for the two-dimensional XY-model (or superfluid), although there is no true long-range order.

- There are no Goldstone modes when the broken symmetry is discrete (e.g. for $n = 1$). In such cases long-range order is possible down to the lower critical dimension of $d_l = 1$.

2.6 Fluctuations, Correlations & Susceptibilities

Our study of Landau mean-field theory showed that the most probable configuration was spatially uniform with $\mathbf{m}(\mathbf{x}) = \bar{m}\hat{\mathbf{e}}_1$, where $\hat{\mathbf{e}}_1$ is a unit vector ($\bar{m}$ is zero for $t > 0$, and equal to $\sqrt{-t/4u}$ for $t < 0$). The role of small fluctuations around such a configuration can be examined by setting

$$\boxed{\mathbf{m}(\mathbf{x}) = [\bar{m} + \phi_l(\mathbf{x})]\,\hat{\mathbf{e}}_1 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}(\mathbf{x})\hat{\mathbf{e}}_\alpha,}$$

where ϕ_l and ϕ_t refer respectively to fluctuations **longitudinal** and **transverse** to the axis of order $\hat{\mathbf{e}}_1$. The transverse fluctuations can take place along any of the $n-1$ directions perpendicular to $\hat{\mathbf{e}}_1$.

After substitution into the Ginzburg-Landau Hamiltonian, a quadratic expansion of the free energy functional with

$$\begin{aligned}(\nabla\mathbf{m})^2 &= (\nabla\phi_l)^2 + (\nabla\phi_t)^2,\\ \mathbf{m}^2 &= \bar{m}^2 + 2\bar{m}\phi_l + \phi_l^2 + \phi_t^2,\\ \mathbf{m}^4 &= \bar{m}^4 + 4\bar{m}^3\phi_l + 6\bar{m}^2\phi_l^2 + 2\bar{m}^2\phi_t^2 + O(\phi_l^3, \phi_l\phi_t^2),\end{aligned}$$

generates the perturbative expansion of the Hamiltonian

$$\begin{aligned}\beta H = V\left(\tfrac{t}{2}\bar{m}^2 + u\bar{m}^4\right) &+ \int d\mathbf{x}\ \left[\tfrac{K}{2}(\nabla\phi_l)^2 + \tfrac{t+12u\bar{m}^2}{2}\phi_l^2\right]\\ &+ \int d\mathbf{x}\ \left[\tfrac{K}{2}(\nabla\phi_t)^2 + \tfrac{t+4u\bar{m}^2}{2}\phi_t^2\right] + O(\phi_l^3, \phi_l\phi_t^2).\end{aligned} \tag{2.20}$$

For spatially uniform fluctuations, one can interpret the prefactors of the quadratic terms in ϕ as “masses” or “restoring forces” (cf. the action of a harmonic oscillator). These effective masses for the fluctuations can be associated with a length scale defined by

$$\begin{aligned}\frac{K}{\xi_l^2} &\equiv t + 12u\bar{m}^2 = \begin{cases} t & t > 0,\\ -2t & t < 0,\end{cases}\\ \frac{K}{\xi_t^2} &\equiv t + 4u\bar{m}^2 = \begin{cases} t & t > 0,\\ 0 & t < 0.\end{cases}\end{aligned} \tag{2.21}$$

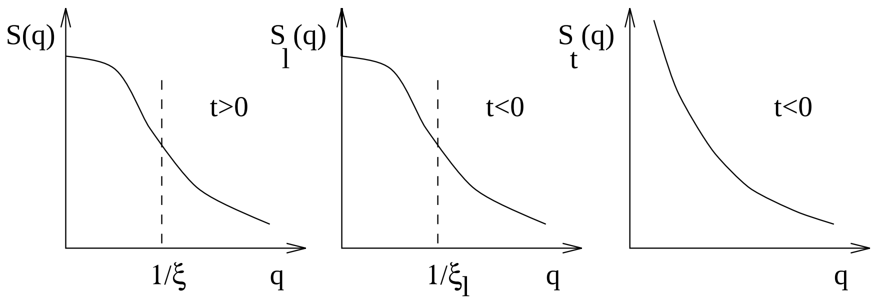

Figure 2.2: Typical neutron scattering amplitude for $t > 0$ and $t < 0$.

(The physical significance of the length scales ξ_l and ξ_t will soon become apparent.) Note that there is no distinction between longitudinal and transverse components in the paramagnetic phase ($t > 0$), while below the transition ($t < 0$), there is no restoring force for the transverse fluctuations (a consequence of the massless Goldstone degrees of freedom discussed previously).

To explore spatial fluctuations and correlation functions, it is convenient to switch to the Fourier representation, wherein the Hamiltonian becomes diagonal (cf. discussion of Gaussian functional integration). After the change of variables

$$\phi(\mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\mathbf{q}}{(2\pi)^d}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \phi(\mathbf{q}),$$

the quadratic Hamiltonian becomes separable into longitudinal and transverse modes,

$$\beta H[\phi_l, \phi_t] = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{K}{2} \left[\left(\mathbf{q}^2 + \xi_l^{-2}\right) |\phi_l(\mathbf{q})|^2 + \left(\mathbf{q}^2 + \xi_t^{-2}\right) |\phi_t(\mathbf{q})|^2\right].$$

Thus, each mode behaves as a Gaussian distributed random variable with zero mean, while the two-point correlation function assumes the form of a **Lorentzian**,

$$\boxed{\langle \phi_\alpha(\mathbf{q})\phi_\beta(\mathbf{q}')\rangle = \delta_{\alpha\beta}\ (2\pi)^d\ \delta^d(\mathbf{q}+\mathbf{q}')\ G_\alpha(\mathbf{q}), \qquad G_\alpha^{-1}(\mathbf{q}) = K(\mathbf{q}^2 + \xi_\alpha^{-2})} \tag{2.22}$$

where the indices α, β denote longitudinal and transverse components. In fact, this equation describing correlations of an order parameter in the vicinity of a critical point was first proposed by **Ornstein and Zernike** as a means to explain the phenomenon of critical opalescence in the light scattering from a fluid in the vicinity of a liquid-gas transition. To understand the mechanism by which ξ sets the characteristic length scale of fluctuations let us consider the scattering amplitude.

In the case of the ferromagnetic model, the two-point correlation function of magnetisation can be observed directly using spin-polarised scattering experiments. The scattering amplitude is related to the Fourier density of scatterers $S(\mathbf{q}) \propto \langle |\mathbf{m}(\mathbf{q})|^2\rangle$ (see Fig. 2.2). The Lorentzian form predicted above usually provides an excellent fit to scattering line shapes away from the critical point. Eq. (2.21) indicates that in the ordered

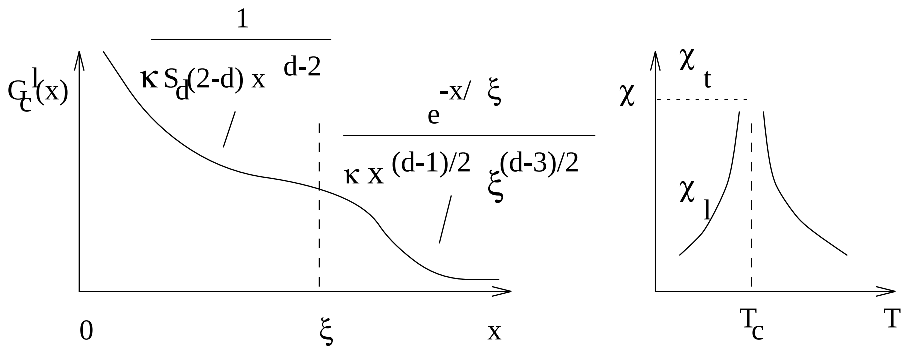

Figure 2.3: Decay of the two-point correlation of magnetisation, and the divergence of the longitudinal and transverse susceptibility in the vicinity of T_c.

phase longitudinal scattering still gives a Lorentzian form (on top of a δ-function at $\mathbf{q} = 0$ due to the spontaneous magnetisation), while transverse scattering always grows as $1/\mathbf{q}^2$. The same power law decay is also predicted to hold at the critical point, $t = 0$. In fact, actual experimental fits yield a power law of the form

$$\boxed{S(\mathbf{q}, T = T_c) \propto \frac{1}{|\mathbf{q}|^{2-\eta}},}$$

with a small positive value of the universal exponent η.

Turning to real space, we find that the average magnetisation is left unaffected by fluctuations, $\langle \phi_\alpha(\mathbf{x})\rangle \equiv \langle m_\alpha(\mathbf{x}) - \bar{m}_\alpha\rangle = 0$, while the connected part of the two-point correlation function takes the form $G^c_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \equiv \langle (m_\alpha(\mathbf{x}) - \bar{m}_\alpha)(m_\beta(\mathbf{x}') - \bar{m}_\beta)\rangle = \langle \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{x}')\rangle$ where

$$\langle \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{x}')\rangle = -\frac{\delta_{\alpha\beta}}{K} I_d(\mathbf{x} - \mathbf{x}', \xi_\alpha), \qquad I_d(\mathbf{x}, \xi) = -\int \frac{d\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \xi^{-2}}. \tag{2.23}$$

The detailed profile of this equation[6] is left as an exercise, but leads to the asymptotics

[6] This Fourier transform is discussed in Chaikin and Lubensky p 156. However, some clue to understanding the form of the transform can be found from the following: Expressed in terms of the modulus q and $d-1$ angles θ_d, the d-dimensional integration measure takes the form

$$d\mathbf{q} = q^{d-1} dq \ \sin^{d-2}\theta_{d-1}\ d\theta_{d-1}\ \sin^{d-3}\theta_{d-2}\ d\theta_{d-2} \cdots d\theta_1,$$

where $0 < \theta_k < \pi$ for $k > 1$, and $0 < \theta_1 < 2\pi$. Thus, by showing that

$$I_d(\mathbf{x}, \xi) = -\frac{1}{(2\pi)^{d/2}|\mathbf{x}|^{d/2-1}} \int_0^{1/a} \frac{q^{d/2} dq}{q^2 + \xi^{-2}} J_{d/2-1}(q|\mathbf{x}|),$$

one can obtain Eq. (2.24) by asymptotic expansion. A second approach is to present the correlator as

$$I_d(\mathbf{x}, \xi) = -\int_0^\infty dt \int \frac{d\mathbf{q}}{(2\pi)^d} e^{i\mathbf{q}\cdot\mathbf{x} - t(\mathbf{q}^2 + \xi^{-2})},$$

integrate over $\mathbf{q}$, and employ a saddle-point approximation.

(see Fig. 2.6)

$$I_d(\mathbf{x},\xi) \simeq \begin{cases} C_d(\mathbf{x}) = \frac{|\mathbf{x}|^{2-d}}{(2-d)S_d} & |\mathbf{x}| \ll \xi, \\ \frac{\xi^{2-d}}{(2-d)S_d} \frac{\exp[-|\mathbf{x}|/\xi]}{|\mathbf{x}/\xi|^{(d-1)/2}} & |\mathbf{x}| \gg \xi. \end{cases} \tag{2.24}$$

From the form of this equation we can interpret the length scale ξ as the **correlation length**.

Using Eq. (2.21) we see that close to the critical point the longitudinal correlation length behaves as

$$\xi_l = \begin{cases} (K/t)^{1/2} & t > 0, \\ (-K/2t)^{1/2} & t < 0. \end{cases}$$

The singularities can be described by $\xi_\pm \simeq \xi_0 B_\pm |t|^{-\nu_\pm}$, where $\nu_\pm = 1/2$ and the ratio $B_+/B_- = \sqrt{2}$ are universal, while $\xi_0 \propto \sqrt{K}$ is not. The transverse correlation length is equivalent to ξ_l for $t > 0$, while it is infinite for all $t < 0$. Eq. (2.24) implies that precisely at T_c, the correlations decay as $1/|\mathbf{x}|^{d-2}$. Again, the experimental decay exponent is usually given by $1/|\mathbf{x}|^{d-2-\eta}$.

These results imply a longitudinal susceptibility of the form (see Fig. 2.6)

$$\chi_l \propto \int d\mathbf{x} \; \overbrace{\langle \phi_l(\mathbf{x})\phi_l(0)\rangle}^{G_l^c(\mathbf{x})} \propto \int_0^{\xi_l} \frac{d\mathbf{x}}{|\mathbf{x}|^{d-2}} \propto \xi_l^2 \simeq A_\pm t^{-1}$$

The universal exponents and amplitude ratios are again recovered from this equation. For $T < T_c$ there is no upper cut-off length for transverse fluctuations, and the divergence of the transverse susceptibility can be related to the system size L, as

$$\chi_t \propto \int d\mathbf{x} \; \overbrace{\langle \phi_l(\mathbf{x})\phi_l(0)\rangle}^{G_t^c(\mathbf{x})} \propto \int_0^{L} \frac{d\mathbf{x}}{|\mathbf{x}|^{d-2}} \propto L^2 \tag{2.25}$$

2.7 Comparison of Theory and Experiment

The validity of the mean-field approximation is assessed in the table below by comparing the results with (approximate) exponents for $d = 3$ from experiment.

Transition type	Material	α	β	γ	ν
		$C \sim \|t\|^{-\alpha}$	$\langle m\rangle \sim \|t\|^{\beta}$	$\chi \sim \|t\|^{-\gamma}$	$\xi \sim \|t\|^{-\nu}$
Ferromag. ($n = 3$)	Fe, Ni	-0.1	0.34	1.4	0.7
Superfluid ($n = 2$)	He^4	0	0.3	1.3	0.7
Liquid-gas ($n = 1$)	CO_2, Xe	0.11	0.32	1.24	0.63
Superconductors		0	1/2	1	1/2
Mean-field		0	1/2	1	1/2

The discrepancy between the mean-field results and experiment signal the failure of the saddle-point approximation. Indeed, the results suggest a dependence of the critical exponents on n (and d). Later we will try to explore ways of going beyond the mean-field approximation. However, before doing so, the experimental results above leave a dilemma. Why do the critical exponents obtained from measurements of the superconducting transition agree so well with mean field theory? Indeed, why do they differ from other transitions which apparently belong to the same universality class? To understand the answer to these questions, it is necessary to examine more carefully the role of fluctuations on the saddle-point.

2.8 Fluctuation Corrections to the Saddle-Point

We are now in a position to determine the corrections to the saddle-point from fluctuations at quadratic order. To do so, it is necessary to determine the fluctuation contribution to the free energy. Applying the matrix (or functional) identity $\ln\det\mathbf{G}^{-1} = -\mathrm{tr}\ln\mathbf{G}$ to Eq. (2.20) we obtain the following estimate for the free energy density

$$f = -\frac{\ln\mathcal{Z}}{V} = \frac{t}{2}\bar{m}^2 + u\bar{m}^4 + \frac{1}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\ \ln[K(\mathbf{q}^2+\xi_l^{-2})] + \frac{n-1}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\ \ln[K(\mathbf{q}^2+\xi_t^{-2})].$$

Inserting the dependence of the correlation lengths on reduced temperature, the singular component of the heat capacity is given by

$$C_{\text{sing.}} \propto -\frac{\partial^2 f}{\partial t^2} = \begin{cases} 0 + \frac{n}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\ \frac{1}{(Kq^2+t)^2} & t>0, \\ \frac{1}{8u} + 2\int\frac{d\mathbf{q}}{(2\pi)^d}\ \frac{1}{(Kq^2-2t)^2} & t<0. \end{cases} \quad (2.26)$$

The behaviour of the integral correction changes dramatically at $d=4$. For $d>4$ the integral diverges at large $\mathbf{q}$ and is dominated by the upper cut-off $\Lambda \approx 1/a$, while for $d<4$, the integral is convergent in both limits. It can be made dimensionless by rescaling $\mathbf{q}$ by ξ^{-1}, and is hence proportional to ξ^{4-d}. Therefore

$$\boxed{\delta C \simeq \frac{1}{K^2}\begin{cases} a^{4-d} & d>4, \\ \xi^{4-d} & d<4. \end{cases}} \quad (2.27)$$

In dimensions $d>4$ fluctuation corrections to the heat capacity add a constant term to the background on each side of the transition. However, the primary form of the discontinuity in $C_{\text{sing.}}$ is unchanged. For $d<4$, the divergence of $\xi \propto t^{-1/2}$ at the transition leads to a correction term that dominates the original discontinuity. Indeed, the correction term suggests an exponent $\alpha = (4-d)/2$. But even this is not reliable — a treatment of higher order corrections will lead to yet more severe divergences. In fact the divergence of δC implies that the conclusions drawn from the saddle-point approximation

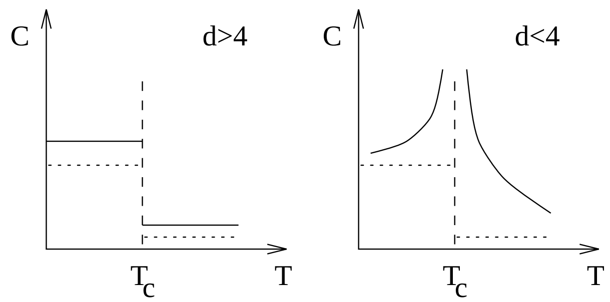

Figure 2.4: Sketch of the heat capacity in the vicinity of the critical point.

are simply no longer reliable in dimensions $d < 4$. One says that Ginzburg-Landau models which belong to this universality class exhibit an **upper critical dimension** d_u of four. Although we reached this conclusion by examining the heat capacity the same conclusion would have been reached for any physical quantity such as magnetisation, or susceptibility.

2.9 Ginzburg Criterion

We have thus established the importance of fluctuations as the probable reason for the failure of the saddle-point approximation to correctly describe the observed exponents. How, therefore, it is possible to account for materials such as superconductors in which the exponents agree well with mean-field theory?

Eq. (2.27) suggests that fluctuations become important when the correlation length begins to diverge. Within the saddle-point approximation, the correlation length diverges as $\xi \simeq \xi_0|t|^{-1/2}$, where $\xi_0 \approx \sqrt{K}$ represents the microscopic length scale. The importance of fluctuations can be assessed by comparing the two terms in Eq. (2.26), the saddle-point discontinuity $\Delta C_{\rm sp} \propto 1/u$, and the correction, δC. Since $K \propto \xi_0^2$, and $\delta C \propto \xi_0^{-d}|t|^{-(4-d)/2}$, fluctuations become important when

$$\boxed{\left(\frac{\xi_0}{a}\right)^{-d} t^{(d-4)/2} \gg \left(\frac{\Delta C_{\rm sp}}{k_B}\right) \Longrightarrow |t| \ll t_G \approx \frac{1}{[(\xi_0/a)^d(\Delta C_{\rm sp}/k_B)]^{2/(4-d)}}.}$$

This inequality is known as the **Ginzburg Criterion**. Naturally, in $d < 4$ it is satisfied sufficiently close to the critical point. However, the resolution of the experiment may not be good enough to get closer than the Ginzburg reduced temperature t_G. If so, the apparent singularities at reduced temperatures $t > t_G$ may show saddle-point behaviour.

In principle, ξ_0 can be deduced experimentally from scattering line shapes. It has to approximately equal the size of the units that undergo ordering at the phase transition. For the liquid-gas transition, ξ_0 can be estimated by $v_c^{1/3}$, where v_c is the critical atomic volume. In superfluids, ξ_0 is approximately equal to the thermal wavelength $\lambda(T)$. Taking $\Delta C_{\rm sp}/k_B \sim 1$ per particle, and $\lambda \sim 2-3$Å we obtain $t_G \sim 10^{-1}-10^{-2}$, a value accessible in

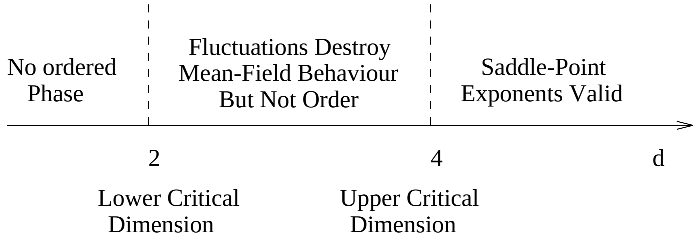

Figure 2.5: Summary of results for the Ginzburg-Landau theory based on mean-field and Gaussian fluctuations.

experiment. However, for a superconductor, the underlying length scale is the separation of Cooper pairs which, as a result of Coulomb repulsion, typically gives $\xi_0 \approx 10^3$Å. This implies $t_G \sim 10^{-16}$, a degree of resolution inaccessible by experiment.

The Ginzburg criterion allows us to restore some credibility to the mean-field theory. As we will shortly see, a theoretical estimate of the critical exponents below the upper critical dimension is typically a challenging endeavour. Yet, for many purposes, a good qualitative understanding of the thermodynamic properties of the experimentally relevant regions of the phase diagram can be understood from the mean-field theory alone.

Self-consistent mean-field: More generally, mean-field theories can be checked for self-consistency as follows. We begin by writing a general Hamiltonian

$$\beta H = \int d\mathbf{x} \int d\mathbf{x}' J(\mathbf{x}-\mathbf{x}') m(\mathbf{x}) m(\mathbf{x}').$$

We can then decompose the field into its mean-field part and the fluctuation part

$$m(\mathbf{x}) = \bar{m} + \phi(\mathbf{x}).$$

In the mean-field approximation second-order fluctuation terms are neglected. This is self-consistent provided

$$\int d\mathbf{x} \int d\mathbf{x}' J(\mathbf{x}-\mathbf{x}') \langle \phi(\mathbf{x}) \phi(\mathbf{x}') \rangle_{\mathrm{MF}} \ll V J \bar{m}^2,$$

where $\int d\mathbf{x} J(\mathbf{x}) = J$.

Because, in the vicinity of the critical point the potential $J(\mathbf{x}-\mathbf{x}')$ is much more short-ranged than the correlation function, we can approximate the left-hand-side by

$$\int d\mathbf{x} \int d\mathbf{x}' J(\mathbf{x}-\mathbf{x}') \langle \phi(\mathbf{x}) \phi(\mathbf{x}') \rangle_{\mathrm{MF}} \approx JV \langle \phi^2(\mathbf{0}) \rangle_{\mathrm{MF}} \propto JV \int \frac{d^d\mathbf{q}}{K\left(\mathbf{q}^2 + \xi^{-2}\right)}.$$

The integral can be split up as follows into a constant and temperature-dependent part

$$\begin{aligned} \frac{JV}{K} \int \frac{d^d\mathbf{q}}{\mathbf{q}^2 + \xi^{-2}} &= \frac{JV}{K} \int \frac{d^d\mathbf{q}}{\mathbf{q}^2} - \frac{JV\xi^{-2}}{K} \int \frac{d^d\mathbf{q}}{\mathbf{q}^2(\mathbf{q}^2 + \xi^{-2})} \\ &= \text{const.} + \frac{JV}{K} \xi^{2-d}. \end{aligned}$$

Since $\bar{m}^2 \propto t \propto K\xi^{-2}$, we conclude that the mean-field approximation is valid when

$$\begin{aligned} \frac{JV}{K}\xi^{2-d} &\ll JVK\xi^{-2} \\ \xi^{4-d} &\ll \xi_0^4, \end{aligned}$$

where $K = \xi_0^2$. For $d < 4$ mean-field theory breaks down when $\xi \sim \xi_0^{\frac{4}{4-d}}$, which for systems with large interaction range ξ_0, such as superconductors, takes place very close to the critical point. Note that the temperature-independent term that we have neglected, although non-singular, might still be large and result in a shift in the critical temperature away from its mean-field value, even when $d > 4$.

2.10 Summary

A summary of our findings for the Ginzburg-Landau Hamiltonian based on mean-field theory and Gaussian fluctuations is shown in Fig. 2.5.