FFLO theory: evaluating the thermodynamic potential

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We start with the definition of the thermodynamic potential Ω in terms of the eigenvalues $\mathcal{G}^{-1}$ of the Nambu-Gorkov Green's function $\boldsymbol{\mathcal{G}}^{-1}$,

$$\Omega = \frac{\Delta^2}{g} - \sum_{\mathcal{G}^{-1}} T \ln\left(1 + \mathrm{e}^{-\frac{1}{T}\mathcal{G}^{-1}}\right), \tag{1}$$

where Δ is the mean-field value of the anomolous average, or equivalently the bosonic Hubbard-Stratonovich transform variable, g is the strength of the contact interaction, T is temperature, and for FFLO the matrix

$$\boldsymbol{\mathcal{G}}^{-1} = \begin{pmatrix} \xi_{\mathbf{k}+\mathbf{q},\uparrow} & \Delta \\ \Delta & -\xi_{-\mathbf{k}+\mathbf{q},\downarrow} \end{pmatrix}, \tag{2}$$

which has eigenvalues

$$\mathcal{G}^{-1} = \xi_- \pm \sqrt{\xi_+^2 + \Delta^2}, \tag{3}$$

where $\xi_\pm = \left(\xi_{\mathbf{k}+\mathbf{q},\uparrow} \pm \xi_{-\mathbf{k}+\mathbf{q},\downarrow}\right)/2$. This means that the thermodynamic potential takes the form

$$\Omega = \frac{\Delta^2}{g} - \sum_{\mathbf{k}} \left(\sqrt{\xi_+^2 + \Delta^2} + \xi_- + T \sum_{\varsigma \in \{+,-\}} \ln\left[1 + \mathrm{e}^{-\frac{1}{T}\left(\sqrt{\xi_+^2+\Delta^2} + \varsigma\xi_-\right)}\right]\right). \tag{4}$$

Noting the simple forms of $\xi_\pm$, $\xi_+ = \frac{1}{2}k^2 + \frac{1}{2}q^2 - \bar{\mu}$ and $\xi_- = \mathbf{k}\cdot\mathbf{q} - \delta\mu \approx k_\mathrm{F} q \cos\theta - \delta\mu$ in a quadratic dispersion, where $\bar{\mu} = (\mu_\uparrow + \mu_\downarrow)/2$ and $\delta\mu = (\mu_\uparrow - \mu_\downarrow)/2$, we may convert the sum over $\mathbf{k}$ in Eq. (4) to an integral, obtaining

$$\Omega = \frac{\Delta^2}{g} - \nu \int \frac{\mathrm{d}\Omega}{4\pi} \int_{\omega_\mathrm{D}}^{\omega_\mathrm{D}} \mathrm{d}\xi_+ \left(\sqrt{\xi_+^2 + \Delta^2} + \xi_- + T \sum_{\varsigma \in \{+,-\}} \ln\left[1 + \mathrm{e}^{-\frac{1}{T}\left(\sqrt{\xi_+^2+\Delta^2} + \varsigma\xi_-\right)}\right]\right), \tag{5}$$

where the density of states $\nu = k_\mathrm{F}/2\pi^2$.

Splitting this integral into two components, the first part may be immediately evaluated as

$$\int \frac{\mathrm{d}\Omega}{4\pi} \int_{\omega_\mathrm{D}}^{\omega_\mathrm{D}} \mathrm{d}\xi_+ \left(\sqrt{\xi_+^2 + \Delta^2} + \xi_- \right) = \omega_\mathrm{D}\sqrt{{\omega_\mathrm{D}}^2 + \Delta^2} + \Delta^2 \mathrm{arcsinh}\frac{\omega_\mathrm{D}}{\Delta} + 2\omega_\mathrm{D}\delta\mu. \tag{6}$$

[b] Name	Condition	θ	ξ_-
E	$qk_\mathrm{F} - \delta\mu < -\Delta$	$[0, \pi]$	$[-qk_\mathrm{F} - \delta\mu, qk_\mathrm{F} - \delta\mu]$
S	$-\Delta < qk_\mathrm{F} - \delta\mu < \Delta$	$\left[\arccos\left(\frac{\delta\mu - \Delta}{qk_\mathrm{F}}\right), \pi\right]$	$[-qk_\mathrm{F} - \delta\mu, -\Delta]$
D	$qk_\mathrm{F} - \delta\mu > \Delta$	$\left[0, \arccos\left(\frac{\delta\mu + \Delta}{qk_\mathrm{F}}\right)\right] \cup \left[\arccos\left(\frac{\delta\mu - \Delta}{qk_\mathrm{F}}\right), \pi\right]$	$[-qk_\mathrm{F} - \delta\mu, -\Delta] \cup [\Delta, qk_\mathrm{F} - \delta\mu]$

Table 1: The blocking regions

The second part, at zero temperature, may be evaluated as

$$
\begin{aligned}
I &= \lim_{T \to 0} \int \frac{\mathrm{d}\Omega}{4\pi} \int_{\omega_\mathrm{D}}^{\omega_\mathrm{D}} \mathrm{d}\xi_+ T \sum_{\varsigma \in \{+,-\}} \ln\left[1 + \mathrm{e}^{-\frac{1}{T}\left(\sqrt{\xi_+^2 + \Delta^2} + \varsigma\xi_-\right)}\right] \\
&= \int \frac{\mathrm{d}\Omega}{4\pi} \int_{\omega_\mathrm{D}}^{\omega_\mathrm{D}} \mathrm{d}\xi_+ \sum_{\varsigma \in \{+,-\}} \left(\varsigma\xi_- - \sqrt{\xi_+^2 + \Delta^2}\right) \Theta\left(\varsigma\xi_- - \sqrt{\xi_+^2 + \Delta^2}\right) \\
&= \int \frac{\mathrm{d}\Omega}{4\pi} \int_{\omega_\mathrm{D}}^{\omega_\mathrm{D}} \mathrm{d}\xi_+ \sum_{\varsigma \in \{+,-\}} \Theta\left(\varsigma\xi_- - \Delta\right) \left[\Theta\left(\sqrt{\xi_-^2 - \Delta^2} - \xi_+\right) - \Theta\left(-\sqrt{\xi_-^2 - \Delta^2} - \xi_+\right)\right] \left(|\xi_-| - \sqrt{\xi_+^2 + \Delta^2}\right) \\
&= \int \frac{\mathrm{d}\Omega}{4\pi} \sum_{\varsigma \in \{+,-\}} \Theta\left(\varsigma\xi_- - \Delta\right) \int_{-\sqrt{\xi_-^2 - \Delta^2}}^{\sqrt{\xi_-^2 - \Delta^2}} \mathrm{d}\xi_+ \left(|\xi_-| - \sqrt{\xi_+^2 + \Delta^2}\right) \\
&= \int \frac{\mathrm{d}\Omega}{4\pi} \sum_{\varsigma \in \{+,-\}} \Theta\left(\varsigma\xi_- - \Delta\right) \left(|\xi_-|\sqrt{\xi_-^2 - \Delta^2} - \Delta^2 \mathrm{arccosh}\frac{|\xi_-|}{\Delta}\right). \qquad (7)
\end{aligned}
$$

where $\Theta(x)$ is the Heaviside step function.

To evaluate the remaining angular integral, we start by noting that we may change variables from the angles (θ, ϕ) to (ξ_-, ϕ), giving $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \frac{\mathrm{d}\xi_- \mathrm{d}\phi}{qk_\mathrm{F}}$. The integration domain over $\xi_\pm$ is indicated in Fig. 1: the blue regions are integrated over in this second integral, and are referred to as the 'blocking regions'. The limits on the integral over ξ_- these blocking regions give rise to are summarised in Table 1.

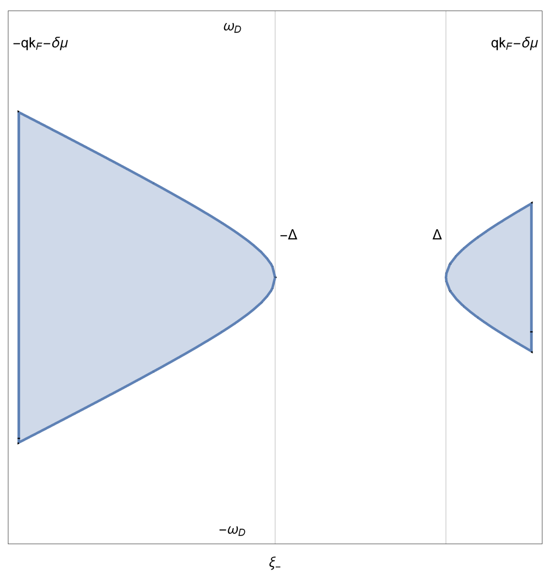

Figure 1: The blocking regions

Heuristically, region E is when qk_F is small compared to $\delta\mu$ such that the blocking region covers the whole angular extent of the integration region, killing off superconductivity. In Fig. 1, the upper limit of integration $qk_\mathrm{F} - \delta\mu$ lies within the left-hand blue region, and the gap and right-hand region do not exist.

Region S is when qk_F is comparable to $\delta\mu$, and the blocking region covers all of the Fermi surface except a 'cap' around the pole. In region S the upper limit of integration lies within the gap between the two blue regions in Fig. 1. When qk_F becomes larger than $\delta\mu$ we enter the region D, where blocking occurs at both poles but leaves a band around the Fermi surface where superconductivity is supported. This is the situation shown in Fig. 1.

Integrating Eq. (7) over the three different regions gives the simple relationships

$$
\begin{aligned}
\mathrm{E}: \quad I = \frac{\Delta^3}{2qk_\mathrm{F}} \Bigg(& f\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) - f\left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) \\
& - \left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) \operatorname{arccosh}\left(-\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) - \left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) \operatorname{arccosh}\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) \Bigg), \\
\mathrm{S}: \quad I = \frac{\Delta^3}{2qk_\mathrm{F}} \Bigg(& f\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) - \left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) \operatorname{arccosh}\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) \Bigg), \\
\mathrm{D}: \quad I = \frac{\Delta^3}{2qk_\mathrm{F}} \Bigg(& f\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) + f\left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) \\
& - \left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) \operatorname{arccosh}\left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) - \left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) \operatorname{arccosh}\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) \Bigg),
\end{aligned} \tag{8}
$$

where the function $f(x) = \frac{1}{3}\left(x^2+2\right)\sqrt{x^2-1}$.

We may now draw everything together, and find the thermodynamic potential (regularised by its value at $\Delta = 0$) as

$$
\begin{aligned}
\Omega(\Delta) - \Omega(0) = \frac{\Delta^2}{g} + \nu \Bigg(& \frac{1}{3}(qk_\mathrm{F})^2 + \delta\mu^2 + {\omega_\mathrm{D}}^2 - \omega_\mathrm{D}\sqrt{{\omega_\mathrm{D}}^2+\Delta^2} - \Delta^2 \operatorname{arcsinh}\frac{\omega_\mathrm{D}}{\Delta} \\
& + \frac{\Delta^3}{2qk_\mathrm{F}} \left(H\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) + H\left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right) \right) \Bigg),
\end{aligned} \tag{9}
$$

where the function

$$
H(x) = \begin{cases} x \operatorname{arccosh} x - \frac{1}{3}(x^2+2)\sqrt{x^2-1}, & x > 1, \\ 0, & |x| < 1, \\ -H(-x), & x < -1. \end{cases} \tag{10}
$$

Using the BCS gap equation $1 = g\nu \operatorname{arcsinh}\frac{\omega_\mathrm{D}}{\Delta} \approx g\nu \ln\frac{2\omega_\mathrm{D}}{\Delta_0}$ we may also eliminate the ultraviolet cutoff ω_D, to give the final form for the FFLO thermodynamic potential,

$$
\frac{\Omega(\Delta)-\Omega(0)}{\nu} = \frac{1}{3}(qk_\mathrm{F})^2 + \delta\mu^2 + \Delta^2\left(\ln\frac{\Delta}{\Delta_0} - \frac{1}{2}\right) + \frac{\Delta^3}{2qk_\mathrm{F}}\left(H\left(\frac{qk_\mathrm{F}+\delta\mu}{\Delta}\right) + H\left(\frac{qk_\mathrm{F}-\delta\mu}{\Delta}\right)\right). \tag{11}
$$