2D FFLO theory: location of second order phase transition

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We can write the thermodynamic potential for FFLO theory as

$$\Omega = \frac{\Delta^2}{g} - \sum_{\mathcal{G}^{-1}} T \ln\left(1 + \mathrm{e}^{-\frac{1}{T}\mathcal{G}^{-1}}\right), \tag{1}$$

where $\mathcal{G}^{-1}$ are the eigenvalues of the Nambu-Forkov Green's function $\boldsymbol{\mathcal{G}}^{-1}$. This matrix takes the form

$$\boldsymbol{\mathcal{G}}^{-1} = \begin{pmatrix} \xi_{\mathbf{k}+\mathbf{q},\uparrow} & \Delta \\ \Delta & -\xi_{-\mathbf{k}+\mathbf{q},\downarrow} \end{pmatrix}, \tag{2}$$

whose eigenvalues may be evaluated to second order in Δ as

$$\mathcal{G}^{-1} = \{\xi_{\mathbf{k}+\mathbf{q},\uparrow} + \frac{\Delta^2}{\xi_{\mathbf{k}+\mathbf{q},\uparrow} + \xi_{-\mathbf{k}+\mathbf{q},\downarrow}} + O(\Delta^4), -\xi_{-\mathbf{k}+\mathbf{q},\downarrow} - \frac{\Delta^2}{\xi_{\mathbf{k}+\mathbf{q},\uparrow} + \xi_{-\mathbf{k}+\mathbf{q},\downarrow}} + O(\Delta^4)\}. \tag{3}$$

Substituting these into the thermodynamic potential, and regularising by the value of Ω at $\Delta = 0$, we find

$$\begin{aligned} \Omega &= \frac{\Delta^2}{g} - \sum_{\mathbf{k}} \left(\frac{\Delta^2}{\xi_{\mathbf{k}+\mathbf{q},\uparrow} + \xi_{-\mathbf{k}+\mathbf{q},\downarrow}} \Theta(\xi_{-\mathbf{k}+\mathbf{q},\downarrow}) - \frac{\Delta^2}{\xi_{\mathbf{k}+\mathbf{q},\uparrow} + \xi_{-\mathbf{k}+\mathbf{q},\downarrow}} \Theta(-\xi_{\mathbf{k}+\mathbf{q},\uparrow}) \right) + O(\Delta^4) \\ &= \frac{\Delta^2}{g} - \sum_{\mathbf{k}} \frac{\Delta^2}{\xi_{\mathbf{k}+\mathbf{q},\uparrow} + \xi_{-\mathbf{k}+\mathbf{q},\downarrow}} \left(1 - \Theta(-\xi_{\mathbf{k}+\mathbf{q},\uparrow}) - \Theta(-\xi_{-\mathbf{k}+\mathbf{q},\downarrow})\right) + O(\Delta^4) \\ &= \frac{\Delta^2}{g} - \sum_{\mathbf{k}} \frac{\Delta^2}{2|\xi_+|} \left(1 - \Theta(\xi_- - |\xi_+|) - \Theta(-\xi_- - |\xi_+|)\right) + O(\Delta^4) \\ &= \left(\frac{1}{g} - \sum_{\mathbf{k} \notin \mathrm{BR}} \frac{1}{2|\xi_+|} \right) \Delta^2 + O(\Delta^4) \\ &= \alpha \Delta^2 + O(\Delta^4), \end{aligned} \tag{4}$$

where $\xi_\pm = (\xi_{\mathbf{k}+\mathbf{q},\uparrow} \pm \xi_{-\mathbf{k}+\mathbf{q},\downarrow})/2$, $\Theta(x)$ is the Heaviside step function, and the region BR is the 'blocking region' shown by the blue region in Fig. 1, in terms of the symmetrized variable ξ_+ and $\cos\theta$.

With a quadratic dispersion, we find the simple relationships $\xi_+ = \frac{1}{2}k^2 + \frac{1}{2}q^2 - \bar{\mu}$ and $\xi_- = \mathbf{k} \cdot \mathbf{q} - \delta\mu \approx k_\mathrm{F} q \cos\theta - \delta\mu$. This allows a simple change of variables from k to ξ_+, giving

$$\begin{aligned} \alpha &= \frac{1}{g} - \nu \int \frac{\mathrm{d}\theta}{2\pi} \int_{|\xi_-|}^{\omega_\mathrm{D}} \frac{\mathrm{d}\xi_+}{\xi_+} \\ &= \frac{1}{g} - \nu \ln \omega_\mathrm{D} + \nu \int \frac{\mathrm{d}\theta}{2\pi} \ln |\xi_-|, \end{aligned} \tag{5}$$

and if we use the BCS gap equation $1 = g\nu \ln \frac{2\omega_\mathrm{D}}{\Delta_0}$ to eliminate g, we find

$$\alpha = \nu \left(\ln \frac{2\omega_\mathrm{D}}{\Delta_0} - \ln \omega_\mathrm{D} + \nu \ln \delta\mu + \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \ln \left| \frac{q k_\mathrm{F}}{\delta\mu} \cos\theta - 1 \right| \right). \tag{6}$$

We evaluate the remaining integral in the box below. Taking the result from there, we get

$$\alpha = \nu \left(\ln \frac{2\delta\mu}{\Delta_0} + \ln \left| \frac{1 + \sqrt{1 - \left(\frac{qk_{\mathrm{F}}}{\delta\mu}\right)^2}}{2} \right| \right), \tag{7}$$

and so the second-order transition to the FFLO state, at $\alpha = 0$, occurs at

$$\frac{2\delta\mu}{\Delta_0} = \left| \frac{2}{1 + \sqrt{1 - \left(\frac{qk_{\mathrm{F}}}{\delta\mu}\right)^2}} \right|, \tag{8}$$

i.e. the maximum $\delta\mu$ where the FFLO state exists is given by

$$\delta\mu = \Delta_0, \tag{9}$$

which corresponds to $qk_{\mathrm{F}} = \delta\mu = \Delta_0$. Compare this to the case in 3D, where the transition occurs at $qk_{\mathrm{F}} = 1.2\delta\mu$.

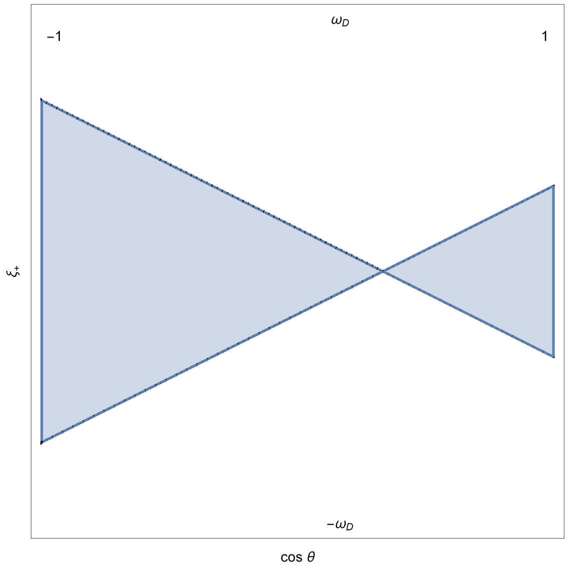

Figure 1: The blocking region, BR

Integral 1 *We wish to evaluate the integral*

$$I(Q) = \int_0^\pi \mathrm{d}\theta \ln|Q\cos\theta - 1|, \qquad Q > 0.$$

First, note that

$$I(Q) = \Re \int_0^\pi \mathrm{d}\theta \ln(Q\cos\theta - 1),$$

and then let us differentiate $I(Q)$ with respect to Q, to obtain

$$I'(Q) = \Re \int_0^\pi \mathrm{d}\theta \frac{1}{Q - \sec\theta}.$$

Changing variables to $u = \sec\theta$, we have

$$I'(Q) = \Re \int_{u\in(-\infty,-1]\cup[1,\infty)} \mathrm{d}u \frac{1}{Q-u}\frac{1}{u^2\sqrt{1-\frac{1}{u^2}}},$$

which is an integral that Mathematica finds quite easy, returning

$$I'(Q) = \Re \begin{cases} \frac{\pi\left(-1+Q+\sqrt{-1+\frac{2}{Q+1}}\right)}{Q(Q-1)}, & 0 < Q < 1, \\ \frac{\pi\left(1+\frac{i}{\sqrt{Q^2-1}}\right)}{Q}, & Q < 1. \end{cases}$$

The integral we were searching for is then

$$I(Q) = \int \mathrm{d}Q I'(Q) = \Re \begin{cases} \pi\ln\left(1+\sqrt{1-Q^2}\right) + C, & 0 < Q < 1, \\ \pi\left(i\arctan\sqrt{Q^2-1} + \ln Q\right) + D, & Q > 1, \end{cases}$$

where the constants C and D are determined by the conditions that $I(0) = 0$ and that $I(Q)$ is continuous across $Q = 1$. These conditions give immediately that $C = -\pi\ln 2$, and so $I(1) = -\pi\ln 2$ and $D = -\pi\ln 2$ as well. Taking the real part, we obtain the solution

$$\begin{aligned} I(Q) &= \begin{cases} \pi\ln\frac{1+\sqrt{1-Q^2}}{2}, & 0 < Q < 1, \\ \pi\ln\frac{Q}{2}, & Q > 1, \end{cases} \\ &= \pi\ln\left|\frac{1+\sqrt{1-Q^2}}{2}\right|. \end{aligned}$$