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Exactly Solvable Hierarchical Optimization Problem Related to Percolation

Thomas M. Fink* and Robin C. Ball
Cavendish Laboratory, University of Cambridge, Cambridge, CB3 0HE, United Kingdom
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We consider a sequence of elementary decisions which must be made in light of successive information learned. A key feature is that the decisions must balance the reduction of immediate cost against learning information and hence securing a wider range of future options—a conflict which motivates us to attach a *value* to information. We analytically derive an optimal decision policy; while each individual decision is elementary, the solution to the collective problem, which may be interpreted as a novel percolation model, exhibits a phase transition and finite size scaling.

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The introduction of probabilistic elements in combinatorial optimization problems (COPs) began with Jaillet [1] in his study of the probabilistic traveling salesman problem (PTSP). The objective is to obtain a first stage solution which minimizes the expected cost of a second stage tour. Bertsimas [2] extended this idea to other probabilistic COPs and suggested the *a priori* optimization heuristic to solve them. These problems are characterized by a stochastically defined instance (in the case of the PTSP, the city locations), which is learned after initially optimizing over all instances and allows further optimization in light of the actual instance (see, e.g., [3]). The inherently *hierarchical* nature of these problems was identified by Kubo and Kasugai [4], who described hierarchical COPs (HCOPs) as follows: they contain two or more levels of decision; precise information about relevant parameters is not known *a priori;* at each successive interval more information becomes available. We propose an HCOP which consists of many elementary decisions but displays nontrivial global behavior and derive an exact analytic solution.

The *constrained* form of the problem, appropriate to the optimal development of a design, is as follows. Decision starts from a unique node at level $N + 1$, from which the costs associated with z descendent nodes at level N are observed. It must be decided how many of these nodes to buy and hence pursue from level N; this process continues in like manner down to level 1. For each of the nodes bought at level n, the price of z descendent nodes at level $n - 1$ are learned. It is then decided which of all of these descendent nodes to buy before proceeding to the next level (see Fig. 1). The objective is to reach at least one node at level 1 with the minimum overall cost. This implies the constraint that at every level at least one node must be bought.

The problem can also be interpreted as one of economic growth, the decision to buy representing investment in future return in the form of negative costs, i.e., profits. In this *economic* form of the problem it is not appropriate to deny the possibility of buying no nodes at some level,

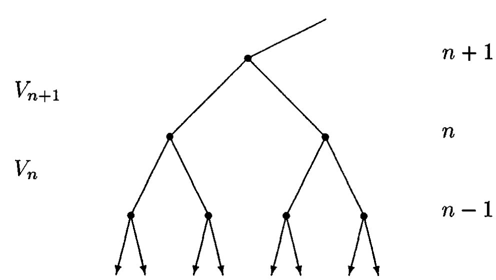

FIG. 1. The decision problem may be summarized as a z-fold Cayley tree (where $z = 2$ in the example shown), N levels deep, with stochastically chosen costs x associated with every node. The tree is traversed from top (level $N + 1$) to bottom (level 1), such that costs on a given level are known (and may thus be purchased) only by paying the cost of the node from which they branch. The objective is to sequentially traverse the tree from top to bottom such that the total incurred cost is minimal.

but of course the result corresponds to termination of the activity.

It is a vital feature of our model that, in either form, the costs are only learned one level at a time and previous decisions cannot be changed. The contrasting case, where all costs are known in advance, would correspond to the problem of directed branched polymers spanning a Cayley tree of sites. We will assume that the costs x are drawn independently from some *a priori* probability distribution such that they may be negative or positive. We concentrate mainly on the case where the distribution is uniform over the finite interval $x \in [\lambda - \frac{1}{2}, \lambda + \frac{1}{2}]$; we argue that the case of a general distribution qualitatively behaves similarly. Unless otherwise stated, we focus on the solution to the economic version of the problem.

Optimality equation.—We set out to obtain the decision policy which minimizes total expected costs, henceforth called the optimal decision policy. We begin by defining V_n to be the expected minimum total cost incurred over $n - 1$ levels, i.e., stemming from a single (purchased) node at level n downwards. That is, $-V_n$ is the expected *value* associated with the subtree stemming from a node x_n, corresponding to the maximum *cost* that we are willing to pay for knowledge of, or access to, that subtree. Identifying $-V_n$ as both the value of a subtree and a bound on costs enables us to recursively define V_{n+1} in terms of the V_n associated with its descendent nodes.

The optimality equation for V_n may be expressed

$$V_{n+1} = \langle \min_a [\mathrm{C}(a(s)) + g(a(s))V_n] \rangle_s , \qquad (1)$$

where $C(a(s))$ is the cost incurred by choosing action a when in state s. The state s is the set of costs observed and the action a is the particular subset of those costs paid. The function $g(a(s))$ gives the number of costs in a.

The optimal policy $a_{\mathrm{opt}}(s)$ is achieved by paying those costs x_n which are less than the maximum cost we are willing to pay, i.e., satisfying $x_n < -V_n$. Summing over the states s, (1) appears as

$$V_{n+1} = \sum_s [P(s)C(a_{\mathrm{opt}}(s)) + P(s)g(a_{\mathrm{opt}}(s))V_n], \quad (2)$$

where $P(s)$ is the probability that state s occurs. The expectation of the cost of a given action is equal to the average cost of a purchased node, c_n, times the expected number of nodes purchased. The optimality equation may then be written

$$V_{n+1} = \sum_s P(s)g(a_{\mathrm{opt}}(s))(c_n + V_n) . \qquad (3)$$

The factor $P(s)g(a_{\mathrm{opt}}(s))$ is the mean number of nodes purchased. With p_{ni} the probability of purchasing the ith node, and noting that the p_{ni} are independent, we may alternatively express the mean number of purchased nodes as the sum of p_{ni} over the z descendent nodes, which yields

$$V_{n+1} = \sum_{i=1}^{z} p_{ni}(c_n + V_n) = zp_n(c_n + V_n) . \qquad (4)$$

Here, p_n is the probability that cost $x_n < -V_n$ and c_n is the mean of x_n given that $x_n < -V_n$, both of which are readily obtained from the cost distribution f_x;

$$p_n = \int_{-\infty}^{-V_n} f_x \, dx, \qquad c_n p_n = q_n = \int_{-\infty}^{-V_n} x f_x \, dx . \qquad (5)$$

The optimality equation may then finally be expressed as

$$V_{n+1} = z(p_n V_n + q_n) . \qquad (6)$$

We have thus derived the recursion relation which governs the optimal decision policy. It is important to note that, while the decision process occurs sequentially going down the tree, the policy is defined recursively going up the tree. This means that the boundary condition is located at the bottom level; since there exist no descendent nodes at level 1, we clearly must have $V_1 = 0$.

The stability of (6) is implied by

$$\left| \frac{dV_{n+1}}{dV_n} \right| < 1 , \qquad (7)$$

which is satisfied for $zp_n < 1$, where zp_n, the number of descendent nodes times the probability that an unknown cost x_n is paid, is the branching rate. Thus the sequence V_n is convergent going up the tree if and only if the mean purchase of subtrees is decaying downwards.

The optimal decision problem may be alternatively expressed, on a given realization of the costs x_n, as sequentially choosing the number of nodes to purchase at each level such that the total incurred cost is minimal. Let B_n be the expected number of costs paid at level n; clearly, $B_n \le z^{N+1-n}$. The total cost incurred at level n, C_n, may then be expressed as $C_n = B_n c_n$. Since

$$B_n = zp_n B_{n+1} \quad \text{and} \quad B_{N+1} = 1 , \qquad (8)$$

we may express the total cost C as

$$C = \sum_{n=1}^{N} C_n = \sum_{n=1}^{N} q_n z^{N+1-n} \prod_{i=n+1}^{N} p_i . \qquad (9)$$

Uniform cost distribution.—The optimal decision policy is dependent on the probability density function, f_x, from which the costs x are chosen, the number of decisions to be made, N, and the degree of the decision tree, z. Here we examine the behavior associated with a uniform distribution, $f_x = 1, x \in [\lambda - \frac{1}{2}, \lambda + \frac{1}{2}]$, for which the optimal policy is most tractable. For convenience, we set the degree of the Cayley tree z = 2.

For $-V_n \le \lambda + \frac{1}{2}$, we can express p_n and q_n from (5) as

$$p_n = -V_n - \lambda + \tfrac{1}{2}, \qquad q_n = \tfrac{1}{2}(V_n^2 - \lambda^2 + \lambda - \tfrac{1}{4}) ; \qquad (10)$$

for $-V_n > \lambda + \frac{1}{2}$, the values of p_n and q_n simplify to 1 and λ, respectively. These physical bounds of p_n (as a probability) and q_n (as a mean) apply to all identities in the remainder of this section. Substitution of (10) into (6) yields the optimality equation in the form of a quadratic map,

$$V_{n+1} = -(V_n + \lambda - \tfrac{1}{2})^2, \qquad V_1 = 0\,. \qquad (11)$$

Equation (11) may be expressed in terms of the more useful parameter p_n, the probability of paying an unknown cost x_n. Eliminating V_n from the left side of (10) and (11) yields

$$p_{n+1} = p_n^2 + \tfrac{1}{2} - \lambda, \qquad p_1 = \tfrac{1}{2} - \lambda\,, \qquad (12)$$

the stable fixed point of which is given by

$$p_f = \tfrac{1}{2} - \sqrt{\lambda - \tfrac{1}{4}}\,, \qquad \lambda \geq \tfrac{1}{4}\,. \qquad (13)$$

Without loss of generality, we restrict our attention to $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$. Observing that (11) and (12) have stable fixed points for $\lambda \geq \frac{1}{4}$ and diverge otherwise, we henceforth refer to the separatrix $\lambda = \frac{1}{4}$ as the critical point λ_c.

First we consider the behavior of the optimal economic policy. We then address the solution of the constrained form of the problem, which, it turns out, coincides with the economic solution for $\lambda < \lambda_c$.

From (8), the growth of B from level n to level $n + 1$ vanishes for $zp_n \leq 1$. Substituting this condition into the left side of (10) implies $-V_n \leq \lambda$, which, it can be shown, is not satisfied for $\lambda \in [\frac{1}{4}, \frac{1}{2}]$. Thus, the branching rate obeys $zp_n < 1$ and the sequence $B_{N+1}, \ldots, B_1$ is decreasing for $\lambda \geq \lambda_c$, corresponding to a region of decreasing economic activity. Moreover, since the B_n scale geometrically with n by the factor zp_n, the tree of purchased nodes is finite in the limit of $N \to \infty$ for $\lambda \geq \lambda_c$.

To examine the behavior of $p(n, \lambda)$, we may approximate the difference equation (12) by a differential equation,

$$\frac{dp}{dn} \simeq p^2 - p + \lambda - \tfrac{1}{2}\,, \qquad (14)$$

provided p_n is slowly varying. For $\lambda \geq \lambda_c$, the solution to (14) appears as

$$p(n, \lambda) = \tfrac{1}{2} - \sqrt{\lambda - \lambda_c}\,\coth\!\Big[\sqrt{\lambda - \lambda_c}\,(n + c)\Big] \quad (15)$$

and approaches its fixed point (13) exponentially. For $\lambda \leq \lambda_c$, it is convenient to express the solution as

$$p(n, \lambda) = \tfrac{1}{2} - \sqrt{\lambda_c - \lambda}\,\cot\!\Big[\sqrt{\lambda_c - \lambda}\,(n + c)\Big], \quad (16)$$

where the constant of integration c is of order unity.

At the critical point $\lambda = \frac{1}{4}$, both of the above reduce to algebraic behavior and p_n slowly approaches its fixed point as

$$p_n \simeq \tfrac{1}{2} - \tfrac{1}{n+c}\,. \qquad (17)$$

When $\lambda = \frac{1}{2}$, there are no negative costs, and thus the obvious optimal policy is to buy zero nodes at all levels; indeed, this is the policy suggested by (15).

With n_0 the level above which all probabilities $p_n = 1$, the optimal decision policy consists of an initial period of maximum growth down to level n_0, during which all nodes are purchased with probability unity. This corresponds to a phase of expansion in which all (positive and negative) costs are paid in the interest of securing future options. Below n_0, the p_n fall below 1, and there is a decrease in purchasing activity such that at the bottom level only negative costs are paid; this may be identified as a profit making regime. Note that since the number of iterations necessary for p_n to exceed 1 is independent of N (for $N > n_0$), the length of the second regime, n_0, is a function of λ only.

We begin our analysis of the constrained policy by examining how well the economic policy, which we have already solved, satisfies the added constraint of reaching the bottom level of the tree. Let b_n be the probability that the subtree of purchased nodes stemming from a single node on level n does not terminate before reaching level 1, i.e., that at least one of the nodes on level 1 is purchased. We may explicitly calculate b_n by noting that it satisfies the recursion relation

$$b_{n+1} = -(p_n b_n)^2 + 2p_n b_n, \qquad b_1 = 1\,, \qquad (18)$$

where p_n is the aforementioned probability of purchasing a node at level n. We are interested in the final term b_{N+1}, the probability that the tree of purchases extends from level $N + 1$ to 1, as a function of λ. Clearly, this function depends on the value of N. As N approaches infinity, $b_{N+1}(\lambda)$ approaches a step function: below λ_c, the probability of purchasing at least one node at the bottom goes to unity, while above λ_c the probability vanishes.

For finite N, the point at which the economic policy ceases to continue to the bottom, and thus at which the economic and constrained policies diverge, occurs not at λ_c but rather $\lambda_{\text{eff}} < \lambda_c$. We are interested in characterizing this divergence point, λ_{eff}, as a function of N. This is most easily addressed in the framework of a percolation model, which we defer until the end.

For $\lambda > \lambda_c$, it is necessary to adapt our economic decision policy such that it does not exhibit terminating behavior. Since the optimal economic policy may be constrained to satisfy the additional constraint of reaching the bottom level of the tree simply by paying the minimum cost available when the economic policy dictates the purchase of none, the optimal constrained policy might be approximated as a concatenation of optimal economic decision policies, each with different initial conditions.

General cost distribution.—For a general cost distribution $f_x = g_x(x - \lambda)$, we may write Eq. (6) as

$$V_{n+1} = z \int_{-\infty}^{-V_n} (V_n + x) g_x\, dx\,. \qquad (19)$$

Change of variables $y = x - \lambda$ and $U_n = -V_n - \lambda$ and integration by parts yields

$$U_{n+1} = -z[(y - U_n)G_y]|_{-\infty}^{U_n} + z\int_{-\infty}^{U_n} G_y\, dy - \lambda\,, \tag{20}$$

where $G_y(y) = \int_{-\infty}^{y} g_y\, dy$ is the cumulative probability distribution. The first term vanishes for all distributions g_y which go to zero more rapidly than y^{-2} as $y \to -\infty$. Accordingly, the optimal policy for a general cost distribution is given by

$$U_{n+1} = z\int_{-\infty}^{U_n}\int_{-\infty}^{y} g_y\, dy\, dy - \lambda = zH(U_n) - \lambda\,. \tag{21}$$

We note that $H > 0$, $H(-\infty) = 0$, $H(y) \to y - c$ as $y \to \infty$, and H is concave upwards. There then exists a critical point λ_c such that $zH(U_n) - \lambda$ intersects the line $U_{n+1} = U_n$ zero times for $\lambda < \lambda_c$, once at $\lambda = \lambda_c$, and twice for $\lambda > \lambda_c$, as shown in Fig. 2. To observe the near critical behavior, we substitute $U_n = u_n + U_c$ in (21), where U_c is the fixed point U_f at λ_c, and expand H about U_c to second order to obtain

$$u_{n+1} + U_c \simeq zH(U_c) + u_n zH'(U_c) + \tfrac{1}{2}u_n^2 zH''(U_c) - \lambda\,. \tag{22}$$

Observing that $zH(U_c) = U_c + \lambda_c$ and $zH'(U_c) = 1$, which determine the critical parameters, and $H''(U_c) = zg(U_c)$, (22) reduces to

$$u_{n+1} \simeq u_n + \tfrac{1}{2}zg(U_c)u_n^2 + \lambda_c - \lambda\,, \tag{23}$$

which generalizes the recursion relation (11) for a general cost distribution and reduces exactly to (11) for $z = 2$ and a uniform distribution.

It is now clear that the optimal policy for any well behaved distribution of costs behaves qualitatively similarly as the case for a uniform distribution, which we have already examined in detail.

Interpretation as a percolation model.—The economic version of the problem may be interpreted as a percolation problem on a Bethe lattice of dimension z. Unlike a conventional percolation problem, the probabilities p are not uniform over the lattice sites, but rather satisfy the recursion relation (12). The analogy between our HOP and a percolation model provides mutual insight into the two seemingly disparate problems.

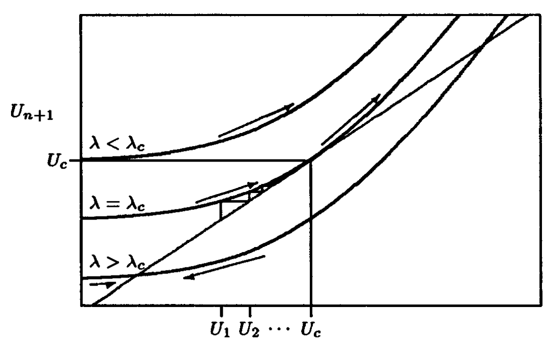

FIG. 2. The function $zH(U_n) - \lambda$ for λ below, at, and above its critical value plotted in $(n + 1, n)$ phase space. The arrows indicate the direction of convergence for increasing n. The sequence $U_n(\lambda)$, $U_1 = -\lambda$, may be obtained geometrically, as shown for $U_n(\lambda_c)$ slowly converging to U_c. For $\lambda < \lambda_c$, the sequence U_n increases to infinity; for $\lambda > \lambda_c$, the sequence U_n converges to $U_f < U_c$.

On a large but finite lattice, conventional percolation models exhibit a shift in the percolation threshold and finite size scaling of various quantities near p_c. We find similar behavior characterized by a shift in λ_c and finite size scaling of properties nearby.

The quantity $b(N + 1, \lambda)$ may be interpreted as the probability of finding a spanning cluster in our Cayley tree of linear dimension N at concentration $p(\lambda)$. The effective critical point $\lambda_{\rm eff}$ at which the cluster spans our finite tree, and consequently percolation occurs, satisfies $b(N + 1, \lambda_{\rm eff}) \simeq \frac{1}{2}$; that is, $\lambda_{\rm eff}$ is the value at which $b_{N+1}(\lambda)$ makes its sharp transition from 0 to 1.

Moreover, we find that

$$\prod_{n=1}^{N} zp_n(\lambda_{\rm eff}) = B_1 \simeq 1\,, \tag{24}$$

and $p_N(\lambda)$ undergoes a sharp transition from $\frac{1}{2}$ to 1 at $\lambda_{\rm eff}$. Imposing the latter condition on the analytic result (16) yields

$$\lambda_c - \lambda_{\rm eff} \simeq (\tfrac{\pi}{N})^2, \tag{25}$$

independent of the precise value of p_N used. Equation (25) can be interpreted as $\lambda_c - \lambda_{\rm eff} \propto N^{-1/\nu}$, where the conventional critical exponent $\nu = \frac{1}{2}$ agrees with ordinary percolation on a Bethe lattice [5].

It would be interesting to explore generalizations of the decision problem to different connectivities of the decisions, i.e., corresponding to different percolation lattices. On a Euclidean lattice, in which the path to a given decision is not unique, we believe our model would correspond to a form of invasion percolation in which the future consequences of each invasion step must be weighed in choosing which step to take.

In conclusion, we have shown that optimizing a sequence of elementary decisions with limited information at each stage yields complex global behavior with a percolationlike critical point. When the mean cost λ lies below a critical threshold λ_c, the optimal number of options to pursue grows exponentially before entering a steady profitable region. Above λ_c, we distinguished between the economic form of the model, in which the tree of options pursued tends to terminate, and the constrained version, in which at least one option must be pursued to the end and for which we can offer only an approximately optimal solution. We demonstrated that our solution is universal in the sense that its qualitative behavior, which we examined in detail for a uniform cost distribution, does not depend on quantitative details of the model, the branching rate z and distribution of costs f_x.

Near the critical point, at which the solution to the two problem versions bifurcates, the economic solution exhibits behavior analogous to a percolation model; we found a finite size shift in λ_c and finite size scaling in the probability b_{N+1} of the solution connecting to the bottom, corresponding to the probability of a spanning cluster in percolation. Our problem naturally maps to a novel percolation model on a Bethe lattice in which the probabilities p of occupation satisfy a recursion relation dependent on z and f_x. The percolation dynamics are characterized by λ; at λ critical the probability of occupation satisfies an inverse radius dependence.

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*Electronic address: tmf20@cus.cam.ac.uk

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