Chapter 8

Critical Exponents, Scaling Relations and an Economic Model

8.1 Critical Exponents and Scaling Relations

Recall the definition of the mean cluster size S,

$$S \propto \sum_{s=1}^{\infty} s^2 n_s, \tag{8.1}$$

which is also known as the second moment of the cluster size distribution. We also defined the cluster strength P as the fraction of arbitrary sites belonging to the infinite cluster; it can be written

$$P \propto \sum_{s=1}^{\infty} s n_s, \tag{8.2}$$

and is the first moment of the cluster size distribution. We argued that P and S (the first and second moments) scaled at the critical point with exponents β and γ, respectively, which are known as critical exponents. There are other critical exponents, such as ν, which we have seen, and α, σ and τ, which we have not. For instance, τ describes the n_s as a function of s, namely,

$$n_s(s) \propto s^{-\tau} e^{-cs}. \tag{8.3}$$

Remarkably, critical exponents do not depend on the lattice structure or the type of percolation (bond or site) but on dimension only. Moreover, they are related according to simple algebraic scaling relations, such as

$$\sigma = \frac{1}{\beta + \gamma} \tag{8.4}$$

and

$$\tau = 2 + \frac{\beta}{\beta + \gamma}. \tag{8.5}$$

In fact, any two critical exponents may be regarded as fundamental, from which all other exponents can be deduced via appropriate scaling relations.

Do we need a new exponent for every moment (as (8.1) and (8.3) suggest)? Fortunately, it can be shown that the divergence of the kth moment M_k, where

$$M_k \propto \sum_{s=1}^{\infty} s^k n_s, \tag{8.6}$$

can be described by the scaling exponent m_k,

$$m_k = \beta - (\beta + \gamma)(k - 1), \tag{8.7}$$

where we recover β for $k = 1$ and γ for $k = 2$. In terms of σ and τ,

$$m_k = \frac{\tau - 1 - k}{\sigma}. \tag{8.8}$$

8.1.1 Fractal dimension in terms of exponents

Recall that the radius of gyration R_s, defined in (7.1), is the typical length scale of a cluster of mass (size) s. It is plausible to relate the radius R_s, mass s and fractal dimension D according to the mass-radius relation,

$$s \propto R_s^D, \tag{8.9}$$

or $R_s \propto s^{1/D}$. We can substitute this into our previous definition of the correlation length ξ,

$$\xi^2 = \frac{\sum_{s=1}^{\infty} R_s^2 s^2 n_s}{\sum_{s=1}^{\infty} s^2 n_s}. \tag{8.10}$$

The denominator of (8.12) is the $k = 2$ moment of the cluster size distribution and, according to (8.8), scales as $\frac{\tau-3}{\sigma}$. The numerator is the $2+\frac{2}{D}$th moment and scales as $\frac{\tau-3-2/D}{\sigma}$. The ratio of the two scales according to $-\frac{2}{D\sigma}$. But recall that the correlation length scales as

$$\xi^2 \propto |p_c - p|^{-2\nu}. \tag{8.11}$$

Matching the two exponents, we find that the fractal dimension D satisfies

$$D = \frac{1}{\sigma\nu}. \tag{8.12}$$

This is very general and powerful result. For two dimensions it gives $D = \frac{91}{48}$, for three $D = 2.53$, for four $D = 3.06$, *etc.*

8.2 An Economic Model

8.2.1 Description of Problem

Consider a regular Cayley tree of coordination number z ($z+1$ if considered as a Bethe lattice). Decision starts from a unique node at level $N+1$, from which the costs associated with z descendant nodes at level N are observed. It must be decided how many of these nodes to buy and hence pursue from level N; this process continues in like manner down to level 1. For each of the nodes bought at level n, the price of z descendant nodes at level $n-1$ are learnt. It is then decided which of all of these descendant nodes to buy before proceeding to the next level (Figure 8.1). The objective is to successively purchase nodes such that the overall cost is minimal in the expected sense.

This problem can be interpreted as one of economic growth, the decision to buy representing investment in future return in the form of negative costs, *i.e.*, profits. In this *economic* model it is not appropriate to deny the possibility of buying no nodes at some level, but of course the result corresponds to termination of the activity.

It is a vital feature of our model that the costs are only learnt one level at a time and previous decisions cannot be changed. We will assume that the costs x are drawn independently from some *a priori* probability distribution such that they may be negative or positive.

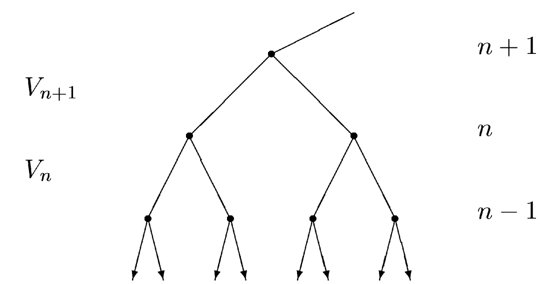

Figure 8.1: The decision problem may be summarised as a z-fold Cayley tree (where $z=2$ in the example shown), N levels deep, with stochasticly chosen costs x associated with every node. The tree is traversed from top (level $N+1$) to bottom (level 1), such that costs on a given level are known (and may thus be purchased) only by paying the cost of the node from which they branch. The objective is to sequentially traverse the tree from top to bottom such that the total incurred cost is minimal.

8.2.2 Optimality Equation

We set out to obtain the decision policy which minimises total expected costs, henceforth called the optimal decision policy. We begin by defining V_n to be the expected minimum total cost incurred over $n-1$ levels, *i.e.*, stemming from a single (purchased) node at level n downwards. That is, $-V_n$ is the expected *value* associated with the subtree stemming from a node x_n, corresponding to the maximum *cost* that we are willing to pay for knowledge of, or access to, that subtree. Identifying $-V_n$ as both the value of a subtree and a bound on costs enables us to recursively define V_{n+1} in terms of the V_n associated with its descendant nodes.

The optimality equation for V_n may be expressed

$$V_{n+1} = \langle \min_a(\mathrm{C}(a(s)) + g(a(s))V_n)\rangle_s, \tag{8.13}$$

where $\mathrm{C}(a(s))$ is the cost incurred by choosing action a when in state

s. The state s is the set of costs observed and the action a is the particular subset of those costs paid. The function $g(a(s))$ gives the number of costs in a.

The optimal policy $a_{opt}(s)$ is achieved by paying those costs x_n which are less than the maximum cost we are willing to pay, *i.e.*, satisfying $x_n < -V_n$. Summing over the states s, (8.13) appears as

$$V_{n+1} = \sum_s (P(s)\mathrm{C}(a_{opt}(s)) + P(s)g(a_{opt}(s))V_n), \tag{8.14}$$

where $P(s)$ is the probability that state s occurs. The expectation of the cost of a given action is equal to the average cost of a purchased node, c_n, times the expected number of nodes purchased. The optimality equation may then be written

$$V_{n+1} = \sum_s P(s)g(a_{opt}(s))(c_n + V_n). \tag{8.15}$$

The factor $P(s)g(a_{opt}(s))$ is the mean number of nodes purchased. With p_{ni} the probability of purchasing the i^{th} node, and noting that the p_{ni} are independent, we may alternatively express the mean number of purchased nodes as the sum of p_{ni} over the z descendant nodes, which yields

$$V_{n+1} = \sum_{i=1}^{z} p_{ni}(c_n + V_n) = zp_n(c_n + V_n). \tag{8.16}$$

Here, p_n is the probability that cost $x_n < -V_n$ and c_n is the mean of x_n given that $x_n < -V_n$, both of which are readily obtained from the cost distribution f_x;

$$p_n = \int_{-\infty}^{-V_n} f_x dx, \qquad c_n p_n = q_n = \int_{-\infty}^{-V_n} x f_x dx. \tag{8.17}$$

The optimality equation may then finally be expressed

$$V_{n+1} = z(p_n V_n + q_n). \tag{8.18}$$

We have thus derived the recursion relation which governs the optimal decision policy. It is important to note that while the decision process occurs sequentially going down the tree, the policy is defined recursively going up the tree. This means that the boundary condition is located at the bottom level; since there exists no descendant nodes at level 1, we clearly must have $V_1 = 0$.

The stability of (8.18) is implied by

$$\left| \frac{dV_{n+1}}{dV_n} \right| < 1, \tag{8.19}$$

which is satisfied for $zp_n < 1$, where zp_n, the number of descendant nodes times the probability than an unknown cost x_n is paid, is the branching rate. Thus the sequence V_n is convergent going up the tree if and only if the mean purchase of subtrees is decaying downwards.

The optimal decision problem may be alternatively expressed, on a given realisation of the costs x_n, as sequentially choosing the number of nodes to purchase at each level such that the total incurred cost is minimal. Let B_n be the expected number of costs paid at level n; clearly, $B_n \leq z^{N+1-n}$. The total cost incurred at level n, C_n, may then be expressed as

$$C_n = B_n c_n. \tag{8.20}$$

Since

$$B_n = zp_n B_{n+1} \quad \text{and} \quad B_{N+1} = 1, \tag{8.21}$$

we may express the total cost C as

$$C = \sum_{n=1}^{N} C_n = \sum_{n=1}^{N} q_n z^{N+1-n} \prod_{i=n+1}^{N} p_i. \tag{8.22}$$

8.2.3 Uniform Cost Distribution

The optimal decision policy is dependent on the probability density function, f_x, from which the costs x are chosen, the number of decisions to be made, N, and the degree of the decision tree, z.

We examine the behaviour associated with a uniform distribution, $f_x = 1, x \in [\lambda - \frac{1}{2}, \lambda + \frac{1}{2}]$, for which the optimal policy is most tractable. For convenience, we set the degree of the Cayley tree $z = 2$.

For $-V_n \leq \lambda + \frac{1}{2}$, we can express p_n and q_n from (8.17) as

$$p_n = -V_n - \lambda + \tfrac{1}{2}, \qquad q_n = \tfrac{1}{2}(V_n^2 - \lambda^2 + \lambda - \tfrac{1}{4}); \tag{8.23}$$

for $-V_n > \lambda + \frac{1}{2}$, the values of p_n and q_n simplify to 1 and λ, respectively. These physical bounds of p_n (as a probability) and q_n (as a mean) apply to all identities in the remainder of this subsection. Substitution of (8.23) into (8.18) yields the optimality equation in the form of a quadratic map,

$$V_{n+1} = -(V_n + \lambda - \tfrac{1}{2})^2, \qquad V_1 = 0. \tag{8.24}$$

Equation (8.24) may be expressed in terms of the more useful parameter p_n, the probability of paying an unknown cost x_n. Eliminating V_n from the left side of (8.23) and (8.24) yields

$$p_{n+1} = p_n^2 + \tfrac{1}{2} - \lambda, \qquad p_1 = \tfrac{1}{2} - \lambda, \tag{8.25}$$

the stable fixed point of which is given by

$$p_f = \tfrac{1}{2} - \sqrt{\lambda - \tfrac{1}{4}}, \qquad \lambda \geq \tfrac{1}{4}. \tag{8.26}$$

Without loss of generality, we restrict our attention to $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$. Observing that (8.24, 8.25) have stable fixed points for $\lambda \geq \frac{1}{4}$ and diverge otherwise, we henceforth refer to the separatrix $\lambda = \frac{1}{4}$ as the critical point λ_c.

From (8.21), the growth of B from level n to level $n+1$ vanishes for $zp_n \leq 1$. Substituting this condition into the left side of (8.23) implies $-V_n \leq \lambda$, which, it can be shown, is not satisfied for $\lambda \in [\frac{1}{4}, \frac{1}{2}]$. Thus, the branching rate obeys $zp_n < 1$ and the sequence $B_{N+1} \ldots B_1$ is decreasing for $\lambda \geq \lambda_c$, corresponding to a region of decreasing economic activity. Moreover, since the B_n scale geometrically with n by the factor zp_n, the tree of purchased nodes is finite in the limit of $N \to \infty$

for $\lambda \geq \lambda_c$.

To examine the behaviour of $p(n, \lambda)$, we may approximate the difference equation (8.25) by a differential equation,

$$\frac{dp}{dn} \simeq p^2 - p + \lambda - \frac{1}{2}, \tag{8.27}$$

provided p_n is slowly varying. For $\lambda \geq \lambda_c$, the solution to (8.27) appears as

$$p(n, \lambda) = \tfrac{1}{2} - \sqrt{\lambda - \lambda_c}\, \coth(\sqrt{\lambda - \lambda_c}\,(n + c)) \tag{8.28}$$

and approaches its fixed point (8.26) exponentially. For $\lambda \leq \lambda_c$, it is convenient to express the solution as

$$p(n, \lambda) = \tfrac{1}{2} - \sqrt{\lambda_c - \lambda}\, \cot(\sqrt{\lambda_c - \lambda}\,(n + c)), \tag{8.29}$$

where the constant of integration c is of order unity.

At the critical point $\lambda = \frac{1}{4}$, both of the above reduce to algebraic behaviour and p_n slowly approaches its fixed point as

$$p_n \simeq \frac{1}{2} - \frac{1}{n + c}. \tag{8.30}$$

When $\lambda = \frac{1}{2}$, there are no negative costs, and thus the obvious optimal policy is to buy zero nodes at all levels; indeed, this is the policy suggested by (8.28).

With n_0 the level above which all probabilities $p_n = 1$, the optimal decision policy consists of an initial period of maximum growth down to level n_0, during which all nodes are purchased with probability unity. This corresponds to a phase of expansion in which all (positive and negative) costs are paid in the interest of securing future options. Below n_0, the p_n fall below 1, and there is a decrease in purchasing activity such that at the bottom level only negative costs are paid; this may be identified as a profit making regime. Note that since the number of iterations necessary for p_n to exceed 1 is independent of N (for $N > n_0$), the length of the second regime, n_0, is a function of λ only.

What is the likelihood of our investment strategy reaching the

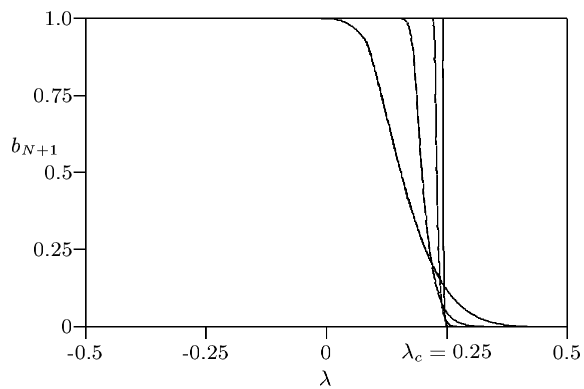

Figure 8.2: Critical transition and finite size effects resulting from the economic policy. Shown is the probability b_{N+1} of the tree of purchased nodes reaching the bottom level as a function of the mean cost λ. Curves are shown, from left to right, for decision tree lengths $N = 4, 8, 16, 32$. For large N, the function approaches a step function, discontinuous at the critical point λ_c.

bottom level of the tree? Let b_n be the probability that the subtree of purchased nodes stemming from a single node on level n does not terminate before reaching level 1, *i.e.*, that at least one of the nodes on level 1 is purchased. We may explicitly calculate b_n by noting that it satisfies the recursion relation

$$b_{n+1} = -(p_n b_n)^2 + 2p_n b_n, \qquad b_1 = 1, \tag{8.31}$$

where p_n is the aforementioned probability of purchasing a node at level n. We are interested in the final term b_{N+1}, the probability that the tree of purchases extends from level $N+1$ to 1, as a function of λ. Clearly, this function depends on the value of N (Figure 8.2). As N approaches infinity, $b_{N+1}(\lambda)$ approaches a step function: below λ_c, the probability of purchasing at least one node at the bottom goes to unity, while above λ_c, the probability vanishes.

For finite N, the point at which the economic policy ceases to con-

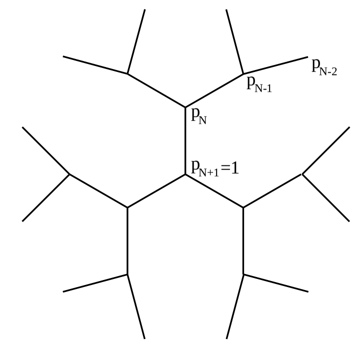

Figure 8.3: The economic problem may be interpreted as percolation on a Bethe lattice with non-uniform probabilities.

tinue to the bottom, and thus at which the economic and constrained policies diverge, occurs not at λ_c but rather $\lambda_{\text{eff}} < \lambda_c$. We are interested in characterising this divergence point, λ_{eff}, as a function of N. This is most easily addressed in the framework of a percolation model, which we consider next.

8.2.4 Interpretation as a Percolation Model

Our economic problem may be interpreted as a percolation problem on a Bethe lattice of dimension z (Figure 8.3). Unlike a conventional percolation problem, the probabilities p are not uniform over the lattice sites, but rather satisfy the recursion relation (8.25).

On a large but finite lattice, conventional percolation models exhibit a shift in the percolation threshold and finite size scaling of various quantities near p_c. We find similar behaviour characterised by a shift in λ_c and finite size scaling of properties nearby.

The quantity $b(N+1, \lambda)$ may be interpreted as the probability of finding a spanning cluster in our Cayley tree of linear dimension N at concentration $p(\lambda)$. The effective critical point λ_{eff} at which the cluster spans our finite tree, and consequently percolation occurs, satisfies $b(N+1, \lambda_{\text{eff}}) \simeq \frac{1}{2}$, that is, λ_{eff} is the value at which $b_{N+1}(\lambda)$

makes its sharp transition from 0 to 1.

Moreover, we find that

$$\prod_{n=1}^{N} z p_n(\lambda_{\text{eff}}) = B_1 \simeq 1. \tag{8.32}$$

and $p_N(\lambda)$ undergoes a sharp transition from $\frac{1}{2}$ to 1 at λ_{eff}. Imposing the latter condition on the analytic result (8.29) yields

$$\lambda_c - \lambda_{\text{eff}} \simeq \left(\frac{\pi}{N}\right)^2 \tag{8.33}$$

independent of the precise value of p_N used. Equation (8.33) can be interpreted as $\lambda_c - \lambda_{\text{eff}} \propto N^{-1/\nu}$, where the conventional critical exponent $\nu = \frac{1}{2}$ agrees with ordinary percolation on a Bethe lattice in Chapter 7.