

Figure 5.2: Oil (black) deposited in porous sediment (white). Repeated lightning strikes leave little of the forest burnt (black).

a very large well does not imply that much oil is accessible. Most of the oil is stored in pores, separated from each other by narrow throats. If a sufficient fraction of the throats are closed (during rock formation or compression), most of the oil is contained in finite isolated clusters. A financially viable well requires a large spanning cluster of open throats, where by large we no longer mean typical cluster radius (as above), but rather its mass, the number of throats (and hence pores).

5.1.2 THE PERCOLATION THRESHOLD

Both of the above are physical examples of percolation, the first site percolation in two dimensions and the second bond percolation in three dimensions. To better understand the physics of percolation, however, it is helpful to consider a less realistic but simpler model.

Consider a finite $L \times L$ square lattice, in which each square can be black (occupied) or white (unoccupied). Each square is independently occupied with probability p . We define a cluster as a group of neighbouring occupied sites, where two sites are neighbours if an

Chapter 5

INTRODUCTION, $D = 1$ AND $D = \infty$

5.1 INTRODUCTION

5.1.1 APPLICATIONS

Consider a large forest, in which trees occupy a fraction p of the land and grass occupies the remaining fraction $1 - p$. The trees are flammable but the grass is not, and a burning tree only ignites neighbours (say, trees within some specified radius). Assume an arbitrary tree is ignited. Clearly, if the forest is sparsely populated, the fire will soon ignite the few neighbouring trees and die out, whereas a dense forest will be largely consumed. How far will the fire spread as a function of the population fraction p ?

Alternatively, consider oil deposits underground. Oil reservoirs usually comprise porous rock saturated by oil, in which the concentration of oil is determined by the porosity of the rock. Unfortunately,

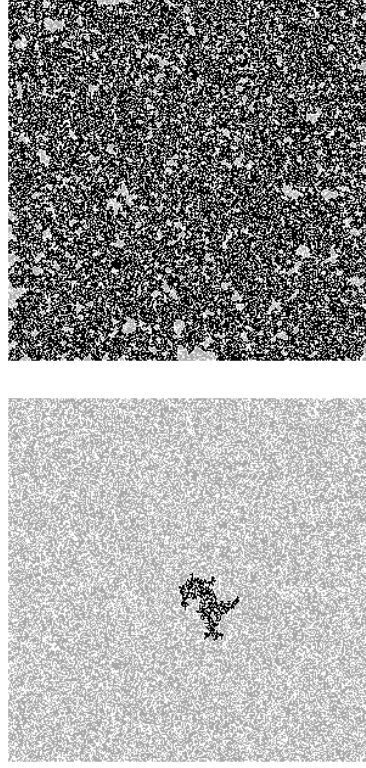


Figure 5.1: Left: In a sparsely populated forest (grey), a fire quickly perishes (black). Right: A dense forest is consumed by fire.

only if they share an edge. A large lattice is likely to have very many such clusters. In short, percolation theory is concerned with the number and property of these clusters as a function of the probability of occupation p .

It is plain that for very low p , clusters are typically small; a single cluster is unlikely to connect opposite edges. When p is very high, the clusters will be large, and are likely to span (or percolate) from one edge to an opposite. Let $\Pi(p, L)$ be the probability that there exists (at least one) spanning cluster. How does Π behave as a function of p and L ?

For small systems (say, a chess board), the probability of a percolating cluster at low p is small but not negligible; likewise, at high p a spanning cluster is likely but not guaranteed. As L increases, the transition from non-spanning to spanning becomes increasingly sharp. Remarkably, on an infinite square lattice Π jumps from zero to one at the fixed critical probability $p_c = 0.5927$ (Figure 5.3).

Such a critical transition is observed on all regular lattices, though the value of p_c varies according to lattice and dimension. Higher dimensional lattices have lower critical probabilities (A $D - 1$ dimensional lattice may be considered a slice through a D dimensional lat-

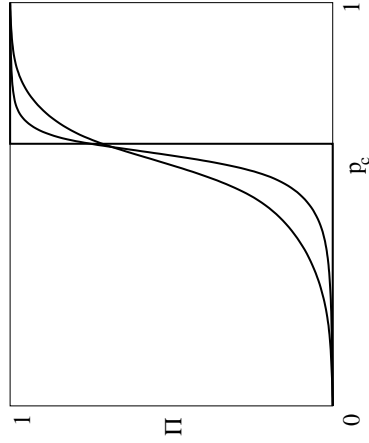


Figure 5.3: A sketch of the probability of a spanning cluster Π as a function of p for (from left to right) $L > 1, L \gg 1, L = \infty$.

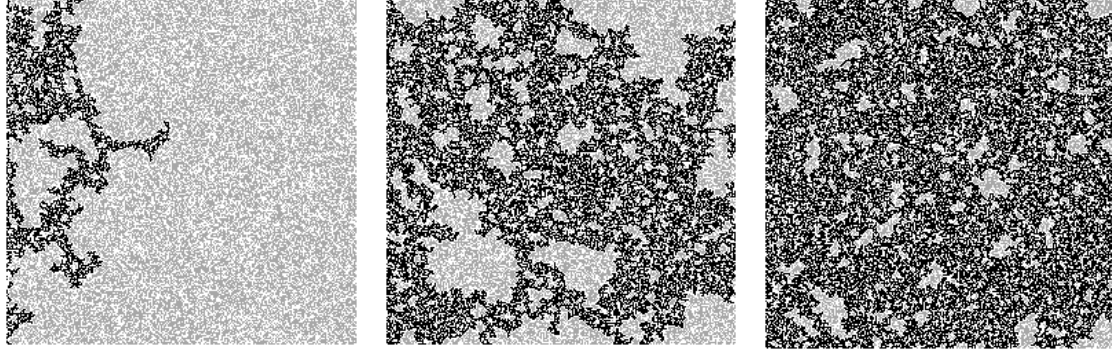


Figure 5.4: 256×256 lattices with probabilities of occupation, from top, $p = p_c - 0.03, p = p_c, p = p_c + .03$.

tice, in which isolated clusters in the $D - 1$ lattice may be connected.) It is worth mentioning here that the above (and forest) model is an example of site-percolation: clusters are defined as collections of neighbouring lattices sites. Percolation may also be defined in terms of clusters of connected bonds (as with the oil model), which can give rise to different values of p_c . We concern ourselves throughout these lectures to site percolation unless otherwise stated.

5.1.3 FURTHER READING

1. Michael E. Fisher, 'Renormalization group theory: Its basis and formulation in statistical physics,' *Rev. Mod. Phys.* **70**, 653 (1998).

Superb discussion of renormalisation. (Only real-space renormalisation will be discussed in these lectures.)

2. Francois Gouyet, *Physics and Fractal Structures* (Springer, New York, 1996).

A physicist's guide to fractals under a variety of guises, including percolation.

3. M. B. Isichenko, 'Percolation, statistical topography and transport in random media,' *Rev. Mod. Phys.* **64**, 961 (1992).

First third provides good 20-page overview of fractals and percolation and links to more detailed papers.

4. Benoit B. Mandelbrot, *The Fractal Geometry of Nature* (W. H. Freeman, New York, 1983).

The well known and rather untechnical (but still insightful) book by the founder of the field.

5. Muhammad Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).

Purely on applications (for theory see below).

6. Dietrich Stauffer and Amnon Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).

Surprisingly clear and readable text, from which parts of these lectures were motivated.

5.2 PERCOLATION IN ONE DIMENSION

Consider site percolation on a 1-dimensional lattice, in which a cluster is defined as an uninterrupted row of sites, and two clusters are separated by one or more sites. The probability of occupation of a site is p .

5.2.1 PERCOLATION THRESHOLD AND NORMALISED CLUSTER NUMBER

For $p = 1$, the entire lattice is occupied and each site belongs to a single infinite cluster. For all $p < 1$, it is plain that an infinite 1-dimensional lattice will possess unoccupied sites, thereby being broken into finite clusters. Accordingly, the percolation threshold in one dimension is

$$p_c = 1. \quad (5.1)$$

What is the probability that, say, four (particular) consecutive sites form a 4-cluster (and are not part of any other larger cluster)? The probability that the four sites are occupied is p^4 , and the probability that the surrounding two sites are unoccupied is $(p - 1)^2$. Thus an arbitrary group of four sites forms a 4-cluster with probability $p^4(1 - p)^2$. More generally, the probability of s -consecutive sites forming an s -cluster is $p^s(1 - p)^2$.

The above quantity may also be considered the probability that an arbitrary site is the left-most member of an s -cluster. The total number of s -clusters, in the limit of large L , is

$$Lp^s(1 - p)^2. \quad (5.2)$$



Figure 5.5: Percolation on a one dimensional lattice.

The number of s -clusters per site, or normalised cluster number n_s , is (5.2) divided by L , that is,

$$n_s = p^s(1 - p)^2. \quad (5.3)$$

5.2.2 MEAN CLUSTER SIZE

What is the mean cluster size? This depends on what we mean by mean (ha!): in the usual sense it would be the average cluster size over a uniform distribution of clusters. Instead, we take the average over a uniform distribution of occupied sites, that is, the mean size of a cluster to which a randomly chosen occupied site belongs. This definition turns out to be the more useful of the two.

The probability that an arbitrary site (occupied or not) belongs to an s -cluster is sn_s and the probability that it belongs to any cluster is $\sum_{s=1}^{\infty} sn_s$. The probability (or weight) w_s that an *occupied* site belongs to an s -cluster is

$$w_s = \frac{sn_s}{\sum_{s=1}^{\infty} sn_s}. \quad (5.4)$$

The mean cluster size is then

$$S = \sum_{s=1}^{\infty} sw_s = \sum_{s=1}^{\infty} \frac{s^2 n_s}{\sum_{s=1}^{\infty} sn_s}. \quad (5.5)$$

The denominator of (5.5) is simply p , since every occupied site belongs to *some* s -cluster. Substituting (5.3), the numerator may be written

$$\begin{aligned} (1 - p)^2 \sum_{s=1}^{\infty} s^2 p^s &= (1 - p)^2 \left(p \sum_s p^s + p^2 \frac{d^2}{dp^2} \sum_s p^s \right) \\ &= p + \frac{2p^2}{1 - p}. \end{aligned} \quad (5.6)$$

Accordingly,

$$S = \frac{1 + p}{1 - p}, \quad (5.7)$$

which tells us that the mean cluster size diverges as the probability of occupation approaches 1, that is, as $p \rightarrow p_c$, which is sensible.

5.3 PERCOLATION ON A BETHE LATTICE

Although the exact solution to percolation in one dimension is straightforward, percolation in two dimensions is more difficult; in fact, the exact solution remains an unsolved problem. Luckily, a less trivial case *can* be solved, *viz.*, percolation on a Bethe lattice.

The Bethe lattice corresponds to an infinitely branching lattice with finite ramification z , the number of daughter branches. The 'origin' has $z + 1$ branches emanating from it and, in the case of an infinite lattice, is indistinguishable from any other site.

5.3.1 PROPERTIES OF A BETHE LATTICE

What does the Bethe lattice have to do with infinite dimensionality? The circumference of a circle in two dimensions is $2\pi r$ and its area is πr^2 . Similarly, the surface area of a sphere is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$. More generally, the surface S of an n -sphere is proportional to r^{n-1} and the volume V to r^n . Accordingly,

$$S \propto V^{1-1/d}. \quad (5.8)$$

The surface of a Bethe lattice grows exponentially with distance (in lattice units) from the origin:

$$S_{\beta}(r) = z(z - 1)^{r-1}. \quad (5.9)$$

Its volume is the sum (over r) of the surface shells:

$$V_{\beta}(r) = 1 + \sum_{i=1}^r z(z - 1)^{r-1} = \frac{z(z - 1)^r}{z - 2} - \frac{2}{z - 2}. \quad (5.10)$$

For all but small r ,

$$S_{\beta}(r) \simeq \frac{z - 1}{z - 2} V_{\beta}(r), \quad (5.11)$$

that is, $S_\beta(r) \propto V_\beta(r)$. From (5.8), this occurs for infinite dimension only.

It can also be shown, for a finite chain in dimension D , that the fraction of paths that it can take on which contains loops vanishes as $D \rightarrow \infty$. The Bethe lattice, by inspection, contains no loops; this feature is crucial to our ability to solve it.

5.3.2 PERCOLATION THRESHOLD

Consider an arbitrary site on an arbitrary cluster. If the cluster is infinite, at least one of the branches connected to the site must lead to infinity. At each step away from the site, we are faced with $z - 1$ new sites, $(z - 1)p$ of which are, on average, occupied. If this number is smaller than unity, the probability of finding an occupied path goes to zero exponentially with distance from the original site. We therefore will find an infinite cluster only if $(1 - z)p \geq 1$, that is,

$$p_c = \frac{1}{z - 1}. \tag{5.12}$$

5.3.3 INFINITE CLUSTER STRENGTH

Above p_c , within an infinite Bethe lattice there exists an infinite cluster. Clearly, as $p \rightarrow p_c$, the infinite cluster occupies an increasing fraction of the total number of sites in the lattice. Call this fraction the cluster strength P ; it is the probability that an arbitrary site belongs to the infinite network.

For convenience set $z = 3$ (Figure 5.6, left). We can calculate P as follows. Let Q be the probability that an arbitrary site is not connected to infinity through a particular branch emanating from that site. Now the probability that a site is not connected to an infinite network (1-P) can occur in two ways: either the site itself is not occupied (with probability $1 - p$) or none of its $z = 3$ branches connects to infinity (with probability Q^3). Accordingly,

$$1 - P = (1 - p) + pQ^3 \tag{5.13}$$

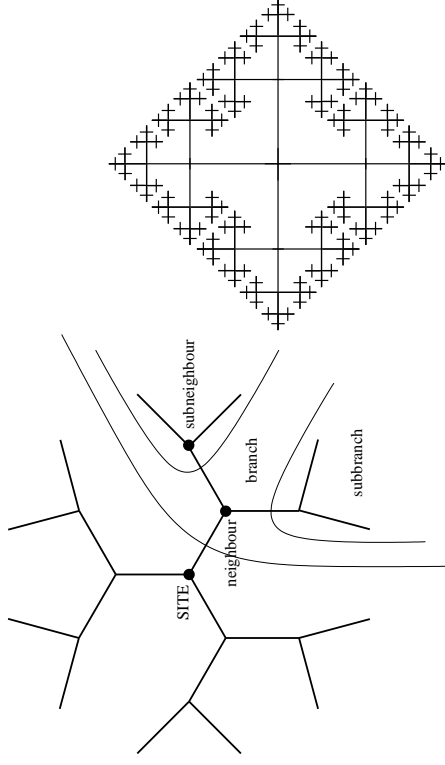


Figure 5.6: Left: A Bethe lattice with coordination number $z = 3$. Although the central site above appears to be the origin, an infinite Bethe lattice is translationally invariant. (The distorted metric is an artifact of embedding the lattice in 2-space.) Right: Another Bethe lattice with $z = 4$.

$$\Rightarrow P = p(1 - Q^3). \tag{5.14}$$

To solve for Q , we note that a branch can not reach infinity in one of two ways: either the neighbour itself is not occupied (with probability $1 - p$), or, if it is occupied, both its adjoining subbranches do not connect to infinity (with probability pQ^2). It follows that

$$Q = (1 - p) + pQ^2, \tag{5.15}$$

which has solutions

$$Q = 1 \quad \text{and} \quad Q = \frac{1 - p}{p}. \tag{5.16}$$

Substituting (5.16) into (5.14) yields

$$P = 0, \quad Q = 1, \tag{5.17}$$

corresponding to the case $p < p_c$, and

$$P = p - \frac{(1-p)^3}{p^2}, \quad Q = (1-p)/p, \quad (5.18)$$

for the case $p > p_c$.

5.3.4 MEAN CLUSTER SIZE S

As in one dimension, the mean cluster strength is the average size of the cluster to which an arbitrary occupied site belongs. Again we set $z = 3$. Let T be the average number of sites within a branch to which a site is connected. The mean cluster size may then be written

$$S = 1 + 3T. \quad (5.19)$$

We can solve for T by noting that, if a neighbour is unoccupied, $T = 0$; if it is occupied, T includes the two connecting subbranches in addition to the occupied site itself. Accordingly,

$$\begin{aligned} T &= (1-p)0 + p(2T+1) & (5.20) \\ \Rightarrow T &= \frac{p}{1-2p}. & (5.21) \end{aligned}$$

Substituting (5.21) into (5.19) yields

$$S = \frac{1+p}{1-2p}. \quad (5.22)$$

The mean cluster size S and the infinite cluster strength P are the order parameters which describe the system below and above threshold. How do they behave in the vicinity of p_c ?

5.3.5 BEHAVIOUR NEAR p_c

Below p_c , (5.22) tells us that as the mean cluster size $S \rightarrow \infty$ as $p \rightarrow p_c$. Expanding S about p_c , we find the leading order behaviour to be

$$S \propto \frac{1}{p_c - p} \equiv (p_c - p)^{-\gamma}, \quad \gamma = 1. \quad (5.23)$$

Just above p_c , there exists a weak infinite cluster, whose leading order behaviour is (from (5.18))

$$P \propto p - p_c \equiv (p - p_c)^\beta, \quad \beta = 1. \quad (5.24)$$

Equations (5.23) and (5.24) are examples of scaling laws: near the critical point quantities of interest behave according to simple geometric relations. Scaling laws are observed in other (finite) dimensions, but there the exponents are not so simple; for instance, in two dimensions P has the same form near p_c but $\beta = \frac{5}{36}$. Remarkably, critical exponents do not depend on the lattice structure or the type of percolation (bond or site) but on dimension only. This is interesting, as it is not so easy for *humans* to give a rigorous definition to lattice dimension.

Equally surprising, for all $d \geq 6$ the various exponents are constant (independent of d) and identical to the Bethe lattice values derived above. The sixth dimension is the so-called upper critical dimension for percolation, with limiting values given by the Bethe lattice (or any other $d \geq 6$).

Other critical exponents will be introduced in later lectures.

$$g_s = \frac{3}{s(2s+1)} \binom{2s+1}{s-1}. \quad (6.5)$$

6.1.2 NORMALISED CLUSTER NUMBERS FOR $D = 2$

What are the cluster numbers for the two dimensional square lattice? The probability that a particular site forms a 2-cluster is

$$n_2 = 2p^2(1-p)^6, \quad (6.6)$$

since 2-clusters require two occupied sites and six unoccupied neighbours, and there are two orientations. The number of 3-clusters is a bit more complicated. There are two configurations, one with two orientations and perimeter 8, the other with four orientations and perimeter 7, whence

$$n_3 = 2p^3(1-p)^8 + 4p^3(1-p)^7. \quad (6.7)$$

Clearly, the perimeter of a cluster is not solely dependent on s , and we must now sum over perimeter t in our definition of n_s ,

$$n_s = \sum_t g_{st} p^s (1-p)^t, \quad (6.8)$$

where g_{st} is the number of lattice cluster, or animals, with size s and perimeter t (this equation is valid generally, not just for $d = 2$). The possible configurations for square lattice animals up to $s = 5$ is shown in Figure 6.1, from which small g_{st} can be calculated. The bad news is that an exact expression for g_{st} has yet to be discovered and, consequently, why non-trivial finite dimensional percolation remains unsolved.

What is known, however, is that the cluster number dies out exponentially for $p < p_c$, but less rapidly for $p > p_c$;

$$\ln n_s \propto -s^\zeta, \quad \text{where } \zeta(p < p_c) = 1, \quad \zeta(p > p_c) = 1 - \frac{1}{d}. \quad (6.9)$$

Chapter 6

CLUSTER STRUCTURE AND FRACTALS

6.1 THE TROUBLE WITH TWO DIMENSIONS

6.1.1 NORMALISED CLUSTER NUMBERS FOR $D = 1$ AND $D = \infty$

Recall that our exact solution for one dimensional percolation depended on our calculation of the normalised cluster numbers, that is, the typical number of clusters of size s per lattice site. For one dimension our result was

$$n_s = p^s(1-p)^2, \quad (6.1)$$

from which we derived the order parameter of the system, the mean cluster size S .

We solved the Bethe lattice by taking advantage of the independence of branches (due to the absence of loops). We could instead have calculated the normalised cluster numbers and calculated the necessary moments of s . For instance, the probability that a particular site on the Bethe lattice (with $z = 3$) is a 1-cluster is

$$n_1 = p(1-p)^3. \quad (6.2)$$

There are three 2-clusters to which a site can belong, but we must divide by 2 so as not to double count:

$$n_2 = 3/2p^2(1-p)^4. \quad (6.3)$$

More generally,

$$n_s = g_s p^s (1-p)^{s+2}, \quad (6.4)$$

where $t = (z-2)s+2$ is the perimeter of an s -cluster on the Bethe lattice and g_s is the number of possible s -clusters per site,

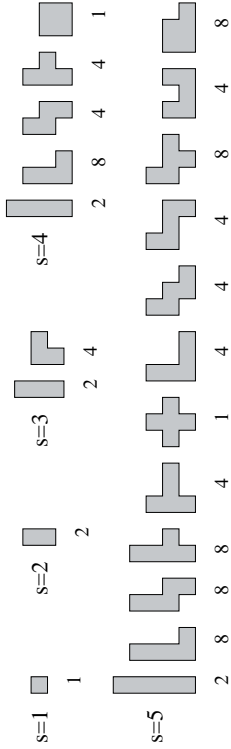


Figure 6.1: Complete configurations for clusters on a square lattice of size 1–5 (lattice animals). Rotations and mirror images of each structure are not shown, but the number of such configurations is listed. The number of lattice animals g_{st} grows rapidly with cluster size, but little is known about these numbers.

6.2 CLUSTER PERIMETER

We have seen that, for the two cases that we are able to solve exactly, the perimeter is proportional to the cluster size s for large s , with constants of proportionality 0 in one dimension and $z - 2$ in two dimensions; both constants may be expressed as $\frac{1-p_c}{p_c}$, with $p_c = 1$ and $\frac{1}{2}$, respectively. How does the perimeter behave in other dimensions? Recall again that the average number of animals of size s per lattice site is

$$n_s = \sum_t g_{st} p^s (1-p)^t. \tag{6.10}$$

Dropping the summation, we have

$$n_s = g_{st} p^s (1-p)^t. \tag{6.11}$$

The ratio $\frac{n_{st}}{n_s}$ is simply the fraction of animals of size s that have perimeter t . We can then write the average perimeter of a cluster of size s as

$$t_s = \sum_t \frac{n_{st} t}{n_s}. \tag{6.12}$$

Let us (out of the blue!) calculate the derivative $\frac{dn_s}{dp}$:

$$\frac{dn_s}{dp} = \frac{s}{p} \sum_t g_{st} p^s (1-p)^t - \frac{1}{1-p} \sum_t t g_{st} p^s (1-p)^t. \tag{6.13}$$

Substituting (6.10) and (6.11) into the left and right sums yields

$$\frac{dn_s}{dp} = \frac{s}{p} n_s - \frac{1}{1-p} \sum_t t n_{st}, \tag{6.14}$$

which, upon applying (6.12), gives

$$\frac{dn_s}{dp} = \frac{s}{p} n_s - \frac{1}{1-p} n_s t_s. \tag{6.15}$$

Rearranging, we have

$$t_s = \frac{1-p}{p} s - (1-p) \frac{d \ln(n_s)}{dp}. \tag{6.16}$$

From (6.9),

$$\frac{d \ln(n_s)}{dp} = c(p) s^{\zeta}, \quad s \rightarrow \infty, \tag{6.17}$$

where c is a p -dependent constant and $\zeta(p < p_c) = 1$, $\zeta(p > p_c) = 1 - \frac{1}{d}$.

Substituting (6.17) into (6.16) yields

$$t_s = \frac{1-p}{p} s + c(p) s^{\zeta}. \tag{6.18}$$

Both above and below p_c the cluster perimeter is proportional to its mass, for all lattices and dimensions. It seems that our lattice animals are very strange beasts indeed.

6.3 CLUSTER DIMENSION

Given the unusual dependence of the cluster perimeter on its mass, we might be led to investigate its typical radius. Usually this is considered the other way round, that is, the mass as a function of radius,

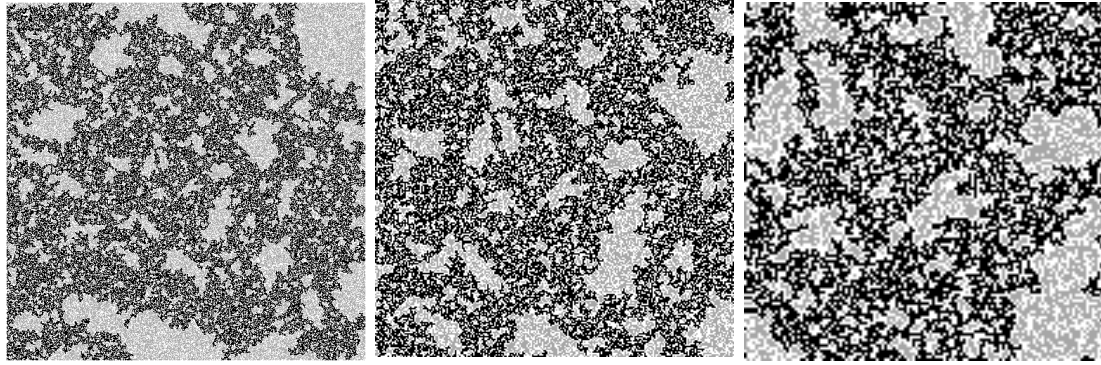


Figure 6.2: An spanning cluster at $p = p_c$ on a 512×512 square lattice (top), magnified to 256×256 (centre) and 128×128 (bottom). Note the statistical self-similarity

which defines an object's dimension. As it turns out, the infinite cluster at p_c is one of the best known examples of a statistical fractal. This can readily be observed by considering a system at p_c under successive magnification (Figure 6.2, $D = 91/48$). The infinite percolating cluster is self-similar in the expected sense, that is, the statistical properties of the infinite cluster are identical on all length scales.

Before we concern ourselves with the fractal geometry of percolation, we'll consider fractal geometry itself.

6.4 FRACTALS

6.4.1 DIMENSION

The usual method of measuring dimension consists of covering the length, area or volume with small (generalised) boxes whose length, area or volume is taken as the usual unit of measurement. In the limit of infinitesimal boxes this is simply the method of multiple integration.

Cantor and others proposed pathological structures to which the above concept of dimension does not yield a sensible measure. One of the simplest is the triadic von Koch curve (1904), which is defined iteratively according to a simple generator (Figure 6.4).

In our discussion of fractal objects we need at our disposal three sorts of dimension. The first is the Euclidian dimension d of the space in which our fractal object F is embedded, and this will always be a positive integer. The second is the topological dimension D_T of F ; D_T is a positive integer and $D_T = 0$ if our fractal object is composed of a set of points, $D_T = 1$ if it is composed of line elements, $D_T = 2$ if it is composed of surface elements, *etc.* For example, the Koch curve has $D_T = 1$ and the Sierpinski carpet (Figure 6.5) $D_T = 2$. The third dimension is the fractal dimension D of F , which need not be an integer and which we set out to define. Both D and $D_T \in [0, d]$, but $D \geq D_T$.

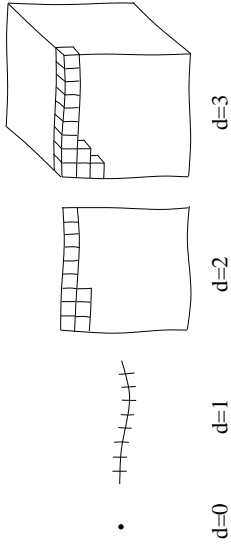


Figure 6.3: Traditional measurement of length, area and volume relies on paving with tiny sticks, squares and cubes.

6.4.2 FRACTAL DIMENSION

The concept of generalised (fractional) dimension was introduced by Hausdorff (1918) and Besicovitch (1929), although Mandelbrot (1975) applied fractional dimension to physical objects and coined the term ‘fractal’.

The Hausdorff definition of (integer) dimension is defined as follows. A d -dimensional Euclidean object is paved with N d -dimensional ‘boxes’ of volume $\mu = \epsilon^d$. When the dimension of the object is unknown, which by assumption is the case, we can instead take $u = \epsilon^\alpha$, where α is unknown. The measure M of the object may be written

$$M = \lim_{\epsilon \rightarrow 0} N \epsilon^\alpha, \tag{6.19}$$

For example, consider a square ($d = 2$) of length L ; the number of boxes of edge ϵ is $N = (L/\epsilon)^2$ and the measure is

$$M = \lim_{\epsilon \rightarrow 0} (L/\epsilon)^2 \epsilon^\alpha = \left(\frac{L^2}{\epsilon^{2-\alpha}} \right). \tag{6.20}$$

For $\alpha = 1$, we find $M = \infty$; the length of a square is infinite. For $\alpha = 3$, $M = 0$; the volume of a square is zero. Only $\alpha = 2$ yields a finite answer for M .

More formally, the α -covering measure is defined

$$M_\alpha(F) = \lim_{\epsilon \rightarrow 0} \inf_{U(F,\epsilon)} \sum_i \epsilon_i^\alpha, \tag{6.21}$$

where F is covered by generalised boxes u_1, u_2, \dots having maximum linear size $\epsilon_1, \epsilon_2, \dots$ and $U(F, \epsilon)$ denotes the set of all possible coverings of F with $\epsilon_i \leq \epsilon$. The Hausdorff-Besicovitch dimension is then defined as

$$D = \inf\{\alpha : M_\alpha(F) = 0\} = \sup\{\alpha : M_\alpha(F) = \infty\}, \tag{6.22}$$

that is, the Hausdorff dimension is that α at which (6.21) jumps from 0 to ∞ , that is, α_{critical} . Remarkably, the Hausdorff dimension can be generalised for non-integer α , which is what we examine below.

6.4.3 FRACTAL DIMENSION OF THE KOCH CURVE

Consider the previously defined Koch curve. As fractals go, it is not very nasty, and we can dispense with minimising over all possible coverings as in (6.21). The measure of the curve is then

$$M = \lim_{n \rightarrow \infty} N \epsilon_n^\alpha, \tag{6.23}$$

where N is the number of boxes of edge ϵ_n necessary to cover the n th-order curve.

We begin by trying $\alpha = 1$. Now the line segment ϵ necessary to

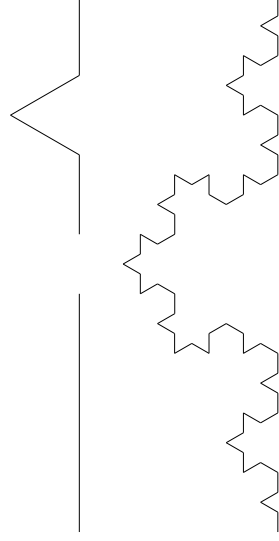


Figure 6.4: The 0th-order, 1st-order and 3rd-order Koch curve. The first two curves comprise its generator. The complete (infinite-order) Koch curve has fractal dimension $D = \frac{\ln 4}{\ln 3}$.

cover each segment of the curve decreases by a factor of three with each iteration of the curve, that is,

$$\epsilon_1 = \frac{L}{3}, \epsilon_2 = \frac{L}{3^2}, \dots, \epsilon_n = \frac{L}{3^n}, \tag{6.24}$$

where L is the linear size of the original (0th-order) curve. The number of segments N grows with n as

$$N_1 = 4, N_2 = 4^2, \dots, N_n = 4^n. \tag{6.25}$$

Accordingly,

$$M = \lim_{n \rightarrow \infty} L \left(\frac{4}{3}\right)^n = \infty. \tag{6.26}$$

Now consider $\alpha = 2$, that is, we cover the curve with increasingly tiny squares of edge ϵ . This time our measuring units are

$$\epsilon_1^2 = \left(\frac{L}{3}\right)^2, \epsilon_2^2 = \left(\frac{L}{3^2}\right)^2, \dots, \epsilon_n^2 = \left(\frac{L}{3^n}\right)^2. \tag{6.27}$$

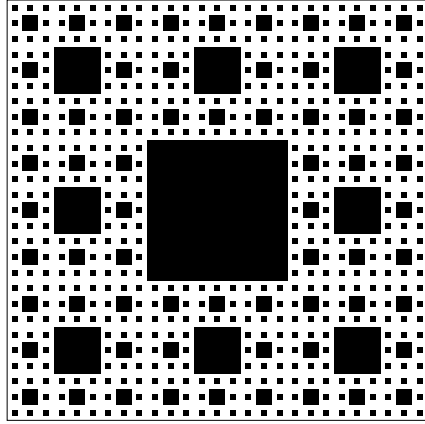


Figure 6.5: A 4th-order Sierpinski carpet, whose limit has fractal dimension $D = \frac{\ln 8}{\ln 3}$.

and the measure is

$$M = \lim_{n \rightarrow \infty} L^2 \left(\frac{4}{9}\right)^n = 0. \tag{6.28}$$

What happens to our ‘boxes’ when α is non-integer? For $\alpha = 1.5$, say, the boxes are of length ϵ but width $\sqrt{\epsilon}$; these boxes have an ever decreasing aspect ratio as $\epsilon \rightarrow 0$. This does not worry us, however, because each need only cover an infinitely thin line. As a repeat of our calculation shows, 1.5 is too big to be the dimension.

The measure can be written for general α as

$$M_\alpha = \lim_{n \rightarrow \infty} L^\alpha \left(\frac{4}{3^\alpha}\right)^n, \tag{6.29}$$

which is finite only for $D = \alpha = \frac{\ln 4}{\ln 3} = 1.262$.

6.4.4 MASS-RADIUS RELATION

The condition that (6.29) give the fractal dimension D is that $\left(\frac{4}{3^\alpha}\right)^n$ be finite in the limit $n \rightarrow \infty$. This gives the mass-radius relation

$$M = c_1(L) L^D, \tag{6.30}$$

where c_1 is a constant in the limit of large L .

The fractal dimension may easily be found by considering its generator. If one iteration produces k elements of linear size $\frac{L}{h}$, we have

$$M(L) = kM(L/h). \tag{6.31}$$

Substituting (6.30) yields

$$c_1(L) L^D = k c_1 \left(\frac{L}{h}\right) \left(\frac{L}{h}\right)^D. \tag{6.32}$$

In the limit of large L , $c_1(L) = c_1 \left(\frac{L}{h}\right)$, and (6.32) can be written

$$D = \frac{\ln k}{\ln h}. \tag{6.33}$$

Let’s apply this to the Sierpinski carpet. At each iteration, 8 elements are generated, each with length reduced by 3. Hence

$$D = \frac{\ln M^n}{\ln L^n} = \frac{\ln 8}{\ln 3} = 1.893. \quad (6.34)$$

Chapter 7

CORRELATION LENGTH, FINITE-SIZE SCALING AND RENORMALISATION

7.1 CORRELATION LENGTH

7.1.1 RADIUS OF GYRATION

The radius of gyration R of a complicated object, and in our case a cluster of size s , is

$$R_s^2 = \frac{1}{s} \sum_{i=1}^s (r_i - \bar{r})^2, \quad (7.1)$$

where

$$\bar{r} = \frac{1}{s} \sum_{i=1}^s r_i. \quad (7.2)$$

We can then write the typical distance between two sites belonging to the same cluster of size s as

$$2R_s^2 = \sum_{i,j=1}^s \frac{|r_i - r_j|^2}{s^2}. \quad (7.3)$$

7.1.2 CORRELATION FUNCTION

The correlation function (or pair connectivity) $g(r)$ is defined as the probability that a site a distance r away from an occupied site belongs to the same cluster. For $r = 0$, the two sites are identical and $g(r)$ is clearly unity. For $r = 1$, $g(r)$ is the probability of occupation p . The probability $g(2)$, apart from one and infinite dimensions, is less clear

due to the presence of loops: there is more than one path by which two non-neighbouring sites can be connected. (The shortest path by itself contributes p^2 .)

The mean number of sites to which an occupied site at the origin is connected is

$$\sum_r g(r), \quad (7.4)$$

where the sum is over all r and hence over all sites. This is simply the mean cluster size S from §5.3.4, that is,

$$\sum_r g(r) = S = \frac{1}{p} \sum_{s=1}^{\infty} s^2 n_s. \quad (7.5)$$

7.1.3 CORRELATION LENGTH

Define the connectivity or correlation length ξ as

$$\xi^2 = \frac{\sum_r r^2 g(r)}{\sum_r g(r)}; \quad (7.6)$$

ξ is some sort of typical distance between two clusters belonging to the same cluster, averaged over all sites. With (7.3) and (7.5), we can write ξ without reference to the unknown $g(r)$ as

$$\xi^2 = \frac{\sum_{s=1}^{\infty} R_s^2 s^2 n_s}{\sum_{s=1}^{\infty} s^2 n_s}. \quad (7.7)$$

What is the correlation length? Unlike the radius of gyration, it is not a measure of the typical linear size of a cluster, since it is an average over all clusters. Stauffer (1992) described it as ‘the radius of those clusters which give the main contribution to the second moment of the cluster size distribution near the percolation threshold.’

At $p \simeq p_c$ on the Bethe lattice, ξ can be expanded about p_c and (not surprisingly) scales as

$$\xi \propto |p_c - p|^{-\nu}, \quad (7.8)$$

where $\nu = \frac{1}{2}$. The exponent ξ cannot be calculated in 2 or more dimensions, but has been strongly conjectured to be $\frac{4}{3}$ in two dimensions.

The correlation length ξ is the only inherent length scale (apart from the lattice spacing) which contributes to percolation structure. Accordingly, the properties of a system exhibit a crossover when examined on scales above and below ξ . For any length scale $L \gg \xi$, the structure of the system is macroscopically homogeneous. When $L \ll \xi$, there exist clusters and all length scales and the spanning cluster appears self-similar, that is to say, is fractal.

7.2 FINITE SIZE SCALING

Recall that, for an infinite system, the probability Π of finding a spanning cluster is 0 for $p < p_c$ and 1 for $p > p_c$. What happens when the lattice is finite? The sharp transition is blurred, that is, Π is small but nonzero at $p \simeq 0$ and likewise $1 - \Pi$ at $p \simeq 1$. Surprisingly, at $p \simeq p_c$, the percolation transition curve does not blur symmetrically about the critical point; for finite systems, there is a downward shifted effective critical probability p_{eff} (Figure 7.1).

Why the asymmetry? As $p \rightarrow p_c$ from below, the correlation length ξ increases according to (7.8). When ξ is comparable to the system size L , percolation becomes likely; the smaller the system size L , the lower the value of p at which $\xi \sim L$.

Replacing ξ with L and p with p_{eff} in (7.8), we find

$$p_c - p_{\text{eff}} \propto L^{-\frac{1}{\nu}}. \quad (7.9)$$

7.3 REAL-SPACE RENORMALISATION

7.3.1 ESTIMATING p_c

Consider coarse-graining an infinite lattice by partitioning it into cells of linear size b . The result is a coarser, renormalised lattice of super-sites, each containing b^d sites. We say that a super-site is occupied (with, say, probability $p'(p)$) if it is spanned by a cluster of sites and unoccupied otherwise.

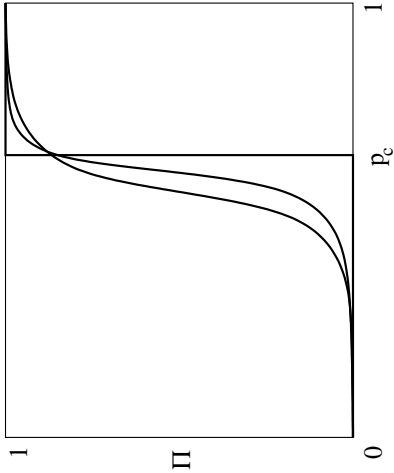


Figure 7.1: The effective percolation probability p_{eff} for finite systems shifts to the left of p_c as the linear system size L is diminished.

At $p = 0$ and $p = 1$, we can be certain that the lattice of renormalised supersites percolates if and only if the lattice of sites percolates, that is, $p' = 0$ and $p' = 1$ when $p = 1$. (The values $p = 0$ and $p = 1$ are called fixed points of the renormalisation transformation.) For $p \in (0, 1)$, however, this need not be the case and, in general, $p' \neq p$.

The key to real-space renormalisation is self-similarity at the critical point $p = p_c$. Since the infinite cluster appears similar on all length scales, the renormalised lattice of supersites must percolate at the same p_c . Thus, $p' = p = p_c$ is also a fixed point of our transformation.

If the cell is not too large, we can write down the renormalised probability of occupation of a supersite $p' = R(p)$ in terms of the original occupation probability p . Our equation for the critical point p_c is then

$$p' = R(p_c) = p_c. \tag{7.10}$$

7.3.2 ESTIMATING ν

Consider now a lattice near the percolation threshold. Recall that, for $p \simeq p_c$, the cluster correlation length ξ satisfies

$$\xi = c_1 |p - p_c|^{-\nu}, \tag{7.11}$$

where c_1 is some constant. Again we partition the lattice by replacing groups of sites of linear size b with a single renormalised supersite. In the renormalised lattice, the correlation length ξ' satisfies

$$\xi' = c_1 |p' - p_c|^{-\nu}, \tag{7.12}$$

where the constant c_1 and the exponent ν are the same as in (7.11) and p' is the concentration of occupied supersites. The correlation length of the renormalised lattice ξ' is in units of the lattice constant b ; in the original lattice units (which we assume here to be 1), we have

$$\xi' = \frac{\xi}{b}. \tag{7.13}$$

Accordingly,

$$b |p' - p_c|^{-\nu} = |p - p_c|^{-\nu}, \tag{7.14}$$

which is the central equation of real-space renormalisation. It follows that

$$\frac{1}{\nu} = \frac{\ln \frac{p' - p_c}{p - p_c}}{\ln b} = \frac{\ln \lambda}{\ln b}, \tag{7.15}$$

where

$$\lambda = \frac{p' - p_c}{p - p_c} = \left. \frac{dp'}{dp} \right|_{p=p_c}. \tag{7.16}$$

7.3.3 RENORMALISATION OF 2D BOND PERCOLATION

We now apply our renormalisation approximation to a real system, namely, two dimensional bond percolation. The original and renormalised lattices are shown in Figure 7.2 (a), and the original and

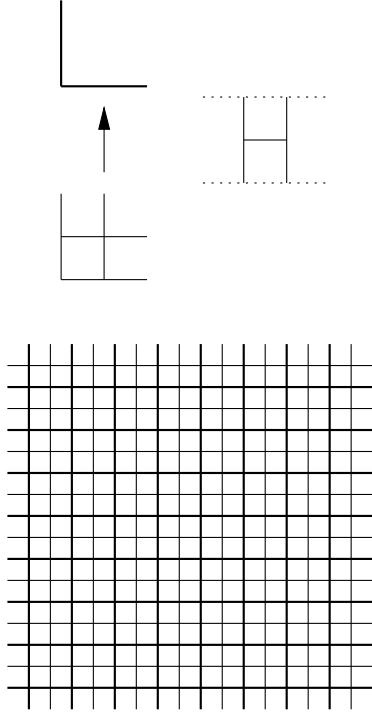
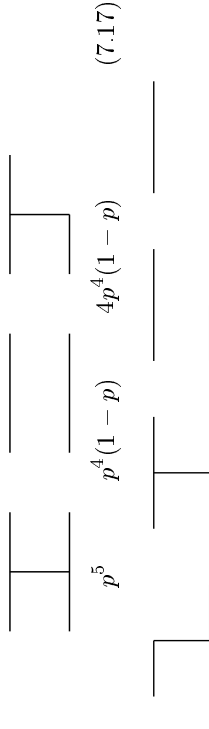


Figure 7.2: Real-space renormalisation for bond percolation on the square lattice. (a) Original 8x8 square lattice, in bold. (b) Original 8x8 bond cell is mapped into renormalised 2x2 bond cell. (c) The cell is occupied if it contains a spanning cluster. Here we consider the horizontal direction only and may thus ignore the two dangling bonds.

renormalised cells in (b). Let's calculate the probability that the original cell is occupied, that is, that it percolates from one edge to the opposite. For definiteness, we choose the horizontal direction, which by inspection cannot make use of the dangling ends. Figure 7.2 (c) shows the five bonds which constitute the cell 'lattice'. What is the probability of a spanning cluster?

The possible spanning clusters and their respective contributions are listed in Figure 7.3. For instance, the probability of the top right configuration is $p^4(1-p)$ and it is 4-fold symmetric. The sum over microstates gives the total probability that a cluster spans the cell, that is, the probability p' that a renormalised cell is occupied. It may be written

$$p' = R(p) = p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + 2p^2(1-p)^3 \quad (7.19)$$



$$2p^3(1-p)^2 \quad 2p^3(1-p)^2 \quad 4p^3(1-p)^2 \quad 2p^2(1-p)^3 \quad (7.18)$$

Figure 7.3: Possible cell configurations which contribute to the occupied probability (*i.e.*, are spanning). Below each configuration is listed the product of its probability and degeneracy.

$$= 2p^5 - 5p^4 + 2p^3 + 2p^2. \quad (7.20)$$

Substituting $p = p' = p^*$ into (7.20) and solving for roots yields $p = 0, \frac{1}{2}$ and 1. The non-trivial solution, $p = \frac{1}{2}$, is our approximation to the percolation threshold p_c ; in this case it happens to be exact. From (7.16),

$$\lambda = \left. \frac{dp'}{dp} \right|_{p=p^*} = \frac{13}{8} \quad (7.21)$$

and

$$\nu = \frac{\ln b}{\ln \lambda} = 1.428. \quad (7.22)$$

The estimate of the critical probability is exact and the estimate of the exponent ν is close to the presumably exact value of $4/3$.

It is worth mentioning that the excellent agreement of our renormalisation approximation with exact results is exceptional for small cells. As the cell size b increases, however, the approximation becomes increasingly accurate.

Chapter 8

CRITICAL EXPONENTS, SCALING RELATIONS AND AN ECONOMIC MODEL

8.1 CRITICAL EXPONENTS AND SCALING RELATIONS

Recall the definition of the mean cluster size S ,

$$S \propto \sum_{s=1}^{\infty} s^2 n_s, \tag{8.1}$$

which is also known as the second moment of the cluster size distribution. We also defined the cluster strength P as the fraction of arbitrary sites belonging to the infinite cluster; it can be written

$$P \propto \sum_{s=1}^{\infty} s n_s, \tag{8.2}$$

and is the first moment of the cluster size distribution. We argued that P and S (the first and second moments) scaled at the critical point with exponents β and γ , respectively, which are known as critical exponents. There are other critical exponents, such as ν , which we have seen, and α, σ and τ , which we have not. For instance, τ describes the n_s as a function of s , namely,

$$n_s(s) \propto s^{-\tau} e^{-cs}. \tag{8.3}$$

Remarkably, critical exponents do not depend on the lattice structure or the type of percolation (bond or site) but on dimension only. Moreover, they are related according to simple algebraic scaling relations, such as

$$\sigma = \frac{1}{\beta + \gamma} \tag{8.4}$$

and

$$\tau = 2 + \frac{\beta}{\beta + \gamma}. \tag{8.5}$$

In fact, any two critical exponents may be regarded as fundamental, from which all other exponents can be deduced via appropriate scaling relations.

Do we need a new exponent for every moment (as (8.1) and (8.3) suggest)? Fortunately, it can be shown that the divergence of the k th moment M_k , where

$$M_k \propto \sum_{s=1}^{\infty} s^k n_s, \tag{8.6}$$

can be described by the scaling exponent m_k ,

$$m_k = \beta - (\beta + \gamma)(k - 1), \tag{8.7}$$

where we recover β for $k = 1$ and γ for $k = 2$. In terms of σ and τ ,

$$m_k = \frac{\tau - 1 - k}{\sigma}. \tag{8.8}$$

8.1.1 FRACTAL DIMENSION IN TERMS OF EXPONENTS

Recall that the radius of gyration R_s , defined in (7.1), is the typical length scale of a cluster of mass (size) s . It is plausible to relate the radius R_s , mass s and fractal dimension D according to the mass-radius relation,

$$s \propto R_s^D, \tag{8.9}$$

or $R_s \propto s^{1/D}$. We can substitute this into our previous definition of the correlation length ξ ,

$$\xi^2 = \frac{\sum_{s=1}^{\infty} R_s^2 s^2 n_s}{\sum_{s=1}^{\infty} s^2 n_s}. \tag{8.10}$$

The denominator of (8.12) is the $k = 2$ moment of the cluster size

distribution and, according to (8.8), scales as $\frac{\tau-3}{\sigma}$. The numerator is the $2 + \frac{2}{D}$ th moment and scales as $\frac{\tau-3-2/D}{\sigma}$. The ratio of the two scales according to $-\frac{2}{D\sigma}$. But recall that the correlation length scales

$$\xi^2 \propto |p_c - p|^{-2\nu}. \tag{8.11}$$

Matching the two exponents, we find that the fractal dimension D satisfies

$$D = \frac{1}{\sigma\nu}. \tag{8.12}$$

This is very general and powerful result. For two dimensions it gives $D = \frac{91}{48}$, for three $D = 2.53$, for four $D = 3.06$, etc.

8.2 AN ECONOMIC MODEL

8.2.1 DESCRIPTION OF PROBLEM

Consider a regular Cayley tree of coordination number $z + 1$ if considered as a Bethe lattice). Decision starts from a unique node at level $N + 1$, from which the costs associated with z descendant nodes at level N are observed. It must be decided how many of these nodes to buy and hence pursue from level N ; this process continues in like manner down to level 1. For each of the nodes bought at level n , the price of z descendant nodes at level $n - 1$ are learnt. It is then decided which of all of these descendant nodes to buy before proceeding to the next level (Figure 8.1). The objective is to successively purchase nodes such that the overall cost is minimal in the expected sense.

This problem can be interpreted as one of economic growth, the decision to buy representing investment in future return in the form of negative costs, *i.e.*, profits. In this *economic* model it is not appropriate to deny the possibility of buying no nodes at some level, but of course the result corresponds to termination of the activity.

It is a vital feature of our model that the costs are only learnt one level at a time and previous decisions cannot be changed. We will assume that the costs x are drawn independently from some *a priori* probability distribution such that they may be negative or positive.

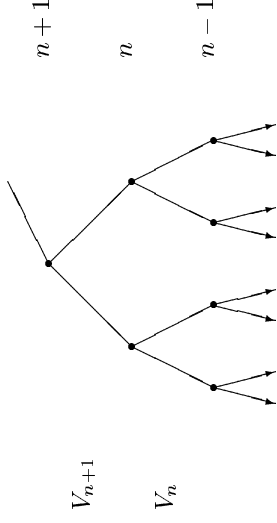


Figure 8.1: The decision problem may be summarised as a z -fold Cayley tree (where $z = 2$ in the example shown), N levels deep, with stochastically chosen costs x associated with every node. The tree is traversed from top (level $N + 1$) to bottom (level 1), such that costs on a given level are known (and may thus be purchased) only by paying the cost of the node from which they branch. The objective is to sequentially traverse the tree from top to bottom such that the total incurred cost is minimal.

8.2.2 OPTIMALITY EQUATION

We set out to obtain the decision policy which minimises total expected costs, henceforth called the optimal decision policy. We begin by defining V_n to be the expected minimum total cost incurred over $n - 1$ levels, *i.e.*, stemming from a single (purchased) node at level n downwards. That is, $-V_n$ is the expected *value* associated with the subtree stemming from a node x_n , corresponding to the maximum *cost* that we are willing to pay for knowledge of, or access to, that subtree. Identifying $-V_n$ as both the value of a subtree and a bound on costs enables us to recursively define V_{n+1} in terms of the V_n associated with its descendant nodes.

The optimality equation for V_n may be expressed

$$V_{n+1} = \langle \min_a (C(a(s)) + g(a(s))V_n) \rangle_s, \tag{8.13}$$

where $C(a(s))$ is the cost incurred by choosing action a when in state

s . The state s is the set of costs observed and the action a is the particular subset of those costs paid. The function $g(a(s))$ gives the number of costs in a .

The optimal policy $a_{opt}(s)$ is achieved by paying those costs x_n which are less than the maximum cost we are willing to pay, *i.e.*, satisfying $x_n < -V_n$. Summing over the states s , (8.13) appears as

$$V_{n+1} = \sum_s (P(s)C(a_{opt}(s)) + P(s)g(a_{opt}(s))V_n), \quad (8.14)$$

where $P(s)$ is the probability that state s occurs. The expectation of the cost of a given action is equal to the average cost of a purchased node, c_n , times the expected number of nodes purchased. The optimality equation may then be written

$$V_{n+1} = \sum_s^z P(s)g(a_{opt}(s))(c_n + V_n). \quad (8.15)$$

The factor $P(s)g(a_{opt}(s))$ is the mean number of nodes purchased. With p_{ni} the probability of purchasing the i^{th} node, and noting that the p_{ni} are independent, we may alternatively express the mean number of purchased nodes as the sum of p_{ni} over the z descendant nodes, which yields

$$V_{n+1} = \sum_{i=1}^z p_{ni}(c_n + V_n) = zp_n(c_n + V_n). \quad (8.16)$$

Here, p_n is the probability that cost $x_n < -V_n$ and c_n is the mean of x_n given that $x_n < -V_n$, both of which are readily obtained from the cost distribution f_x ;

$$p_n = \int_{-\infty}^{-V_n} f_x dx, \quad c_n p_n = q_n = \int_{-\infty}^{-V_n} x f_x dx. \quad (8.17)$$

The optimality equation may then finally be expressed

$$V_{n+1} = z(p_n V_n + q_n). \quad (8.18)$$

We have thus derived the recursion relation which governs the optimal decision policy. It is important to note that while the decision process occurs sequentially going down the tree, the policy is defined recursively going up the tree. This means that the boundary condition is located at the bottom level; since there exists no descendant nodes at level 1, we clearly must have $V_1 = 0$.

The stability of (8.18) is implied by

$$\left| \frac{dV_{n+1}}{dV_n} \right| < 1, \quad (8.19)$$

which is satisfied for $zp_n < 1$, where zp_n , the number of descendant nodes times the probability that an unknown cost x_n is paid, is the branching rate. Thus the sequence V_n is convergent going up the tree if and only if the mean purchase of subtrees is decaying downwards.

The optimal decision problem may be alternatively expressed, on a given realisation of the costs x_n , as sequentially choosing the number of nodes to purchase at each level such that the total incurred cost is minimal. Let B_n be the expected number of costs paid at level n ; clearly, $B_n \leq z^{N+1-n}$. The total cost incurred at level n , C_n , may then be expressed as

$$C_n = B_n c_n. \quad (8.20)$$

Since

$$B_n = zp_n B_{n+1} \quad \text{and} \quad B_{N+1} = 1, \quad (8.21)$$

we may express the total cost C as

$$C = \sum_{n=1}^N C_n = \sum_{n=1}^N q_n z^{N+1-n} \prod_{i=n+1}^N p_i. \quad (8.22)$$

8.2.3 UNIFORM COST DISTRIBUTION

The optimal decision policy is dependent on the probability density function, f_x , from which the costs x are chosen, the number of decisions to be made, N , and the degree of the decision tree, z .

We examine the behaviour associated with a uniform distribution, $f_x = 1, x \in [\lambda - \frac{1}{2}, \lambda + \frac{1}{2}]$, for which the optimal policy is most tractable. For convenience, we set the degree of the Cayley tree $z = 2$.

For $-V_n \leq \lambda + \frac{1}{2}$, we can express p_n and q_n from (8.17) as

$$p_n = -V_n - \lambda + \frac{1}{2}, \quad q_n = \frac{1}{2}(V_n^2 - \lambda^2 + \lambda - \frac{1}{4}); \quad (8.23)$$

for $-V_n > \lambda + \frac{1}{2}$, the values of p_n and q_n simplify to 1 and λ , respectively. These physical bounds of p_n (as a probability) and q_n (as a mean) apply to all identities in the remainder of this subsection. Substitution of (8.23) into (8.18) yields the optimality equation in the form of a quadratic map,

$$V_{n+1} = -(V_n + \lambda - \frac{1}{2})^2, \quad V_1 = 0. \quad (8.24)$$

Equation (8.24) may be expressed in terms of the more useful parameter p_n , the probability of paying an unknown cost x_n . Eliminating V_n from the left side of (8.23) and (8.24) yields

$$p_{n+1} = p_n^2 + \frac{1}{2} - \lambda, \quad p_1 = \frac{1}{2} - \lambda, \quad (8.25)$$

the stable fixed point of which is given by

$$p_f = \frac{1}{2} - \sqrt{\lambda - \frac{1}{4}}, \quad \lambda \geq \frac{1}{4}. \quad (8.26)$$

Without loss of generality, we restrict our attention to $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$. Observing that (8.24, 8.25) have stable fixed points for $\lambda \geq \frac{1}{4}$ and diverge otherwise, we henceforth refer to the separatrix $\lambda = \frac{1}{4}$ as the critical point λ_c .

From (8.21), the growth of B from level n to level $n+1$ vanishes for $zp_n \leq 1$. Substituting this condition into the left side of (8.23) implies $-V_n \leq \lambda$, which, it can be shown, is not satisfied for $\lambda \in [\frac{1}{4}, \frac{1}{2}]$. Thus, the branching rate obeys $zp_n < 1$ and the sequence $B_{N+1} \dots B_1$ is decreasing for $\lambda \geq \lambda_c$, corresponding to a region of decreasing economic activity. Moreover, since the B_n scale geometrically with n by the factor zp_n , the tree of purchased nodes is finite in the limit of $N \rightarrow \infty$

for $\lambda \geq \lambda_c$.

To examine the behaviour of $p(n, \lambda)$, we may approximate the difference equation (8.25) by a differential equation,

$$\frac{dp}{dn} \simeq p^2 - p + \lambda - \frac{1}{2}, \quad (8.27)$$

provided p_n is slowly varying. For $\lambda \geq \lambda_c$, the solution to (8.27) appears as

$$p(n, \lambda) = \frac{1}{2} - \sqrt{\lambda - \lambda_c} \coth(\sqrt{\lambda - \lambda_c}(n + c)) \quad (8.28)$$

and approaches its fixed point (8.26) exponentially. For $\lambda \leq \lambda_c$, it is convenient to express the solution as

$$p(n, \lambda) = \frac{1}{2} - \sqrt{\lambda_c - \lambda} \cot(\sqrt{\lambda_c - \lambda}(n + c)), \quad (8.29)$$

where the constant of integration c is of order unity.

At the critical point $\lambda = \frac{1}{4}$, both of the above reduce to algebraic behaviour and p_n slowly approaches its fixed point as

$$p_n \simeq \frac{1}{2} - \frac{1}{n + c}. \quad (8.30)$$

When $\lambda = \frac{1}{2}$, there are no negative costs, and thus the obvious optimal policy is to buy zero nodes at all levels; indeed, this is the policy suggested by (8.28).

With n_0 the level above which all probabilities $p_n = 1$, the optimal decision policy consists of an initial period of maximum growth down to level n_0 , during which all nodes are purchased with probability unity. This corresponds to a phase of expansion in which all (positive and negative) costs are paid in the interest of securing future options. Below n_0 , the p_n fall below 1, and there is a decrease in purchasing activity such that at the bottom level only negative costs are paid; this may be identified as a profit making regime. Note that since the number of iterations necessary for p_n to exceed 1 is independent of N (for $N > n_0$), the length of the second regime, n_0 , is a function of λ only.

What is the likelihood of our investment strategy reaching the

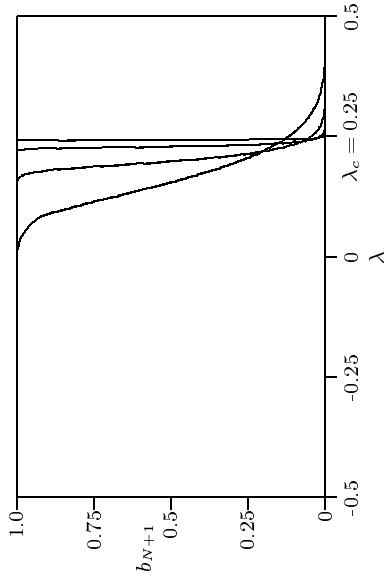


Figure 8.2: Critical transition and finite size effects resulting from the economic policy. Shown is the probability b_{N+1} of the tree of purchased nodes reaching the bottom level as a function of the mean cost λ . Curves are shown, from left to right, for decision tree lengths $N = 4, 8, 16, 32$. For large N , the function approaches a step function, discontinuous at the critical point λ_c .

bottom level of the tree? Let b_n be the probability that the subtree of purchased nodes stemming from a single node on level n does not terminate before reaching level 1, *i.e.*, that at least one of the nodes on level 1 is purchased. We may explicitly calculate b_n by noting that it satisfies the recursion relation

$$b_{n+1} = -(p_n b_n)^2 + 2p_n b_n, \quad b_1 = 1, \quad (8.31)$$

where p_n is the aforementioned probability of purchasing a node at level n . We are interested in the final term b_{N+1} , the probability that the tree of purchases extends from level $N + 1$ to 1, as a function of λ . Clearly, this function depends on the value of N (Figure 8.2). As N approaches infinity, $b_{N+1}(\lambda)$ approaches a step function: below λ_c , the probability of purchasing at least one node at the bottom goes to unity, while above λ_c , the probability vanishes.

For finite N , the point at which the economic policy ceases to con-

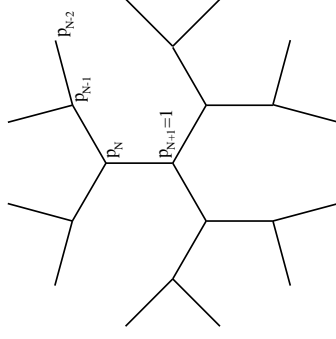


Figure 8.3: The economic problem may be interpreted as percolation on a Bethe lattice with non-uniform probabilities.

tinue to the bottom, and thus at which the economic and constrained policies diverge, occurs not at λ_c but rather $\lambda_{\text{eff}} < \lambda_c$. We are interested in characterising this divergence point, λ_{eff} , as a function of N . This is most easily addressed in the framework of a percolation model, which we consider next.

8.2.4 INTERPRETATION AS A PERCOLATION MODEL

Our economic problem may be interpreted as a percolation problem on a Bethe lattice of dimension z (Figure 8.3). Unlike a conventional percolation problem, the probabilities p are not uniform over the lattice sites, but rather satisfy the recursion relation (8.25).

On a large but finite lattice, conventional percolation models exhibit a shift in the percolation threshold and finite size scaling of various quantities near p_c . We find similar behaviour characterised by a shift in λ_c and finite size scaling of properties nearby.

The quantity $b(N + 1, \lambda)$ may be interpreted as the probability of finding a spanning cluster in our Cayley tree of linear dimension N at concentration $p(\lambda)$. The effective critical point λ_{eff} at which the cluster spans our finite tree, and consequently percolation occurs, satisfies $b(N + 1, \lambda_{\text{eff}}) \simeq \frac{1}{2}$, that is, λ_{eff} is the value at which $b_{N+1}(\lambda)$

makes its sharp transition from 0 to 1.

Moreover, we find that

$$\prod_{n=1}^N zp_n(\lambda_{\text{eff}}) = B_1 \simeq 1. \quad (8.32)$$

and $p_N(\lambda)$ undergoes a sharp transition from $\frac{1}{2}$ to 1 at λ_{eff} . Imposing the latter condition on the analytic result (8.29) yields

$$\lambda_c - \lambda_{\text{eff}} \simeq \left(\frac{\pi}{N}\right)^2 \quad (8.33)$$

independent of the precise value of p_N used. Equation (8.33) can be interpreted as $\lambda_c - \lambda_{\text{eff}} \propto N^{-1/\nu}$, where the conventional critical exponent $\nu = \frac{1}{2}$ agrees with ordinary percolation on a Bethe lattice in Chapter 7.