

Figure 5.2: Oil (black) deposited in porous sediment (white). Repeated lightning strikes leave little of the forest burnt (black).

a very large well does not imply that much oil is accessible. Most of the oil is stored in pores, separated from each other by narrow throats. If a sufficient fraction of the throats are closed (during rock formation or compression), most of the oil is contained in finite isolated clusters. A financially viable well requires a large spanning cluster of open throats, where by large we no longer mean typical cluster radius (as above), but rather its mass, the number of throats (and hence pores).

5.1.2 THE PERCOLATION THRESHOLD

Both of the above are physical examples of percolation, the first site percolation in two dimensions and the second bond percolation in three dimensions. To better understand the physics of percolation, however, it is helpful to consider a less realistic but simpler model.

Consider a finite $L \times L$ square lattice, in which each square can be black (occupied) or white (unoccupied). Each square is independently occupied with probability p . We define a cluster as a group of neighbouring occupied sites, where two sites are neighbours if an

Chapter 5

INTRODUCTION, $D = 1$ AND $D = \infty$

5.1 INTRODUCTION

5.1.1 APPLICATIONS

Consider a large forest, in which trees occupy a fraction p of the land and grass occupies the remaining fraction $1 - p$. The trees are flammable but the grass is not, and a burning tree only ignites neighbours (say, trees within some specified radius). Assume an arbitrary tree is ignited. Clearly, if the forest is sparsely populated, the fire will soon ignite the few neighbouring trees and die out, whereas a dense forest will be largely consumed. How far will the fire spread as a function of the population fraction p ?

Alternatively, consider oil deposits underground. Oil reservoirs usually comprise porous rock saturated by oil, in which the concentration of oil is determined by the porosity of the rock. Unfortunately,

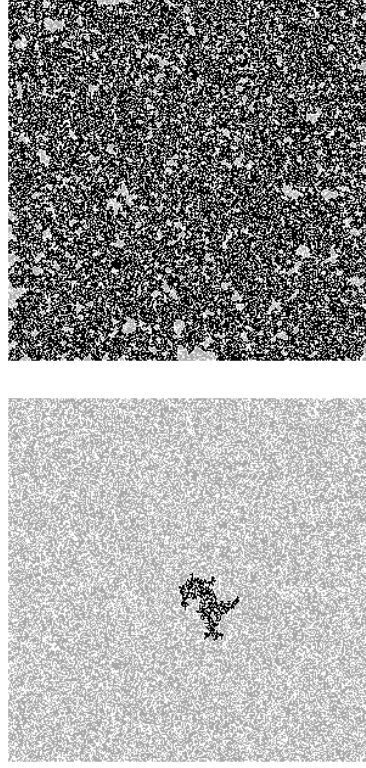


Figure 5.1: Left: In a sparsely populated forest (grey), a fire quickly perishes (black). Right: A dense forest is consumed by fire.

only if they share an edge. A large lattice is likely to have very many such clusters. In short, percolation theory is concerned with the number and property of these clusters as a function of the probability of occupation p .

It is plain that for very low p , clusters are typically small; a single cluster is unlikely to connect opposite edges. When p is very high, the clusters will be large, and are likely to span (or percolate) from one edge to an opposite. Let $\Pi(p, L)$ be the probability that there exists (at least one) spanning cluster. How does Π behave as a function of p and L ?

For small systems (say, a chess board), the probability of a percolating cluster at low p is small but not negligible; likewise, at high p a spanning cluster is likely but not guaranteed. As L increases, the transition from non-spanning to spanning becomes increasingly sharp. Remarkably, on an infinite square lattice Π jumps from zero to one at the fixed critical probability $p_c = 0.5927$ (Figure 5.3).

Such a critical transition is observed on all regular lattices, though the value of p_c varies according to lattice and dimension. Higher dimensional lattices have lower critical probabilities (A $D - 1$ dimensional lattice may be considered a slice through a D dimensional lat-

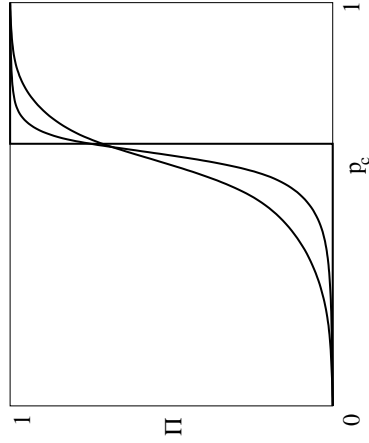


Figure 5.3: A sketch of the probability of a spanning cluster Π as a function of p for (from left to right) $L > 1, L \gg 1, L = \infty$.

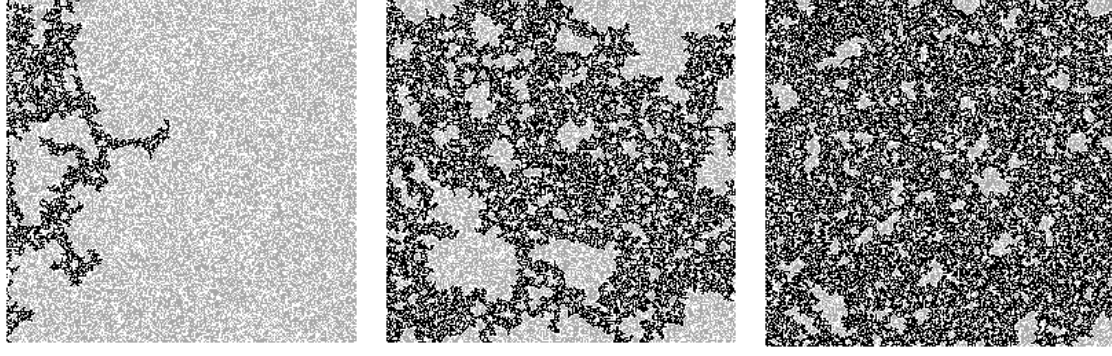


Figure 5.4: 256×256 lattices with probabilities of occupation, from top, $p = p_c - 0.03, p = p_c, p = p_c + .03$.

tice, in which isolated clusters in the $D - 1$ lattice may be connected.) It is worth mentioning here that the above (and forest) model is an example of site-percolation: clusters are defined as collections of neighbouring lattices sites. Percolation may also be defined in terms of clusters of connected bonds (as with the oil model), which can give rise to different values of p_c . We concern ourselves throughout these lectures to site percolation unless otherwise stated.

5.1.3 FURTHER READING

1. Michael E. Fisher, 'Renormalization group theory: Its basis and formulation in statistical physics,' *Rev. Mod. Phys.* **70**, 653 (1998).

Superb discussion of renormalisation. (Only real-space renormalisation will be discussed in these lectures.)

2. Francois Gouyet, *Physics and Fractal Structures* (Springer, New York, 1996).

A physicist's guide to fractals under a variety of guises, including percolation.

3. M. B. Isichenko, 'Percolation, statistical topography and transport in random media,' *Rev. Mod. Phys.* **64**, 961 (1992).

First third provides good 20-page overview of fractals and percolation and links to more detailed papers.

4. Benoit B. Mandelbrot, *The Fractal Geometry of Nature* (W. H. Freeman, New York, 1983).

The well known and rather untechnical (but still insightful) book by the founder of the field.

5. Muhammad Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).

Purely on applications (for theory see below).

6. Dietrich Stauffer and Annon Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).

Surprisingly clear and readable text, from which parts of these lectures were motivated.

5.2 PERCOLATION IN ONE DIMENSION

Consider site percolation on a 1-dimensional lattice, in which a cluster is defined as an uninterrupted row of sites, and two clusters are separated by one or more sites. The probability of occupation of a site is p .

5.2.1 PERCOLATION THRESHOLD AND NORMALISED CLUSTER NUMBER

For $p = 1$, the entire lattice is occupied and each site belongs to a single infinite cluster. For all $p < 1$, it is plain that an infinite 1-dimensional lattice will possess unoccupied sites, thereby being broken into finite clusters. Accordingly, the percolation threshold in one dimension is

$$p_c = 1. \quad (5.1)$$

What is the probability that, say, four (particular) consecutive sites form a 4-cluster (and are not part of any other larger cluster)? The probability that the four sites are occupied is p^4 , and the probability that the surrounding two sites are unoccupied is $(p - 1)^2$. Thus an arbitrary group of four sites forms a 4-cluster with probability $p^4(1 - p)^2$. More generally, the probability of s -consecutive sites forming an s -cluster is $p^s(1 - p)^2$.

The above quantity may also be considered the probability that an arbitrary site is the left-most member of an s -cluster. The total number of s -clusters, in the limit of large L , is

$$Lp^s(1 - p)^2. \quad (5.2)$$



Figure 5.5: Percolation on a one dimensional lattice.

The number of s -clusters per site, or normalised cluster number n_s , is (5.2) divided by L , that is,

$$n_s = p^s(1 - p)^2. \tag{5.3}$$

5.2.2 MEAN CLUSTER SIZE

What is the mean cluster size? This depends on what we mean by mean (ha!): in the usual sense it would be the average cluster size over a uniform distribution of clusters. Instead, we take the average over a uniform distribution of occupied sites, that is, the mean size of a cluster to which a randomly chosen occupied site belongs. This definition turns out to be the more useful of the two.

The probability that an arbitrary site (occupied or not) belongs to an s -cluster is sn_s and the probability that it belongs to any cluster is $\sum_{s=1}^{\infty} sn_s$. The probability (or weight) w_s that an *occupied* site belongs to an s -cluster is

$$w_s = \frac{sn_s}{\sum_{s=1}^{\infty} sn_s}. \tag{5.4}$$

The mean cluster size is then

$$S = \sum_{s=1}^{\infty} sw_s = \sum_{s=1}^{\infty} \frac{s^2 n_s}{\sum_{s=1}^{\infty} sn_s}. \tag{5.5}$$

The denominator of (5.5) is simply p , since every occupied site belongs to *some* s -cluster. Substituting (5.3), the numerator may be written

$$\begin{aligned} (1 - p)^2 \sum_{s=1}^{\infty} s^2 p^s &= (1 - p)^2 \left(p \frac{d}{dp} \sum_s p^s + p^2 \frac{d^2}{dp^2} \sum_s p^s \right) \\ &= p + \frac{2p^2}{1 - p}. \end{aligned} \tag{5.6}$$

Accordingly,

$$S = \frac{1 + p}{1 - p}, \tag{5.7}$$

which tells us that the mean cluster size diverges as the probability of occupation approaches 1, that is, as $p \rightarrow p_c$, which is sensible.

5.3 PERCOLATION ON A BETHE LATTICE

Although the exact solution to percolation in one dimension is straightforward, percolation in two dimensions is more difficult; in fact, the exact solution remains an unsolved problem. Luckily, a less trivial case *can* be solved, *viz.*, percolation on a Bethe lattice.

The Bethe lattice corresponds to an infinitely branching lattice with finite ramification z , the number of daughter branches. The 'origin' has $z + 1$ branches emanating from it and, in the case of an infinite lattice, is indistinguishable from any other site.

5.3.1 PROPERTIES OF A BETHE LATTICE

What does the Bethe lattice have to do with infinite dimensionality?

The circumference of a circle in two dimensions is $2\pi r$ and its area is πr^2 . Similarly, the surface area of a sphere is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$. More generally, the surface S of an n -sphere is proportional to r^{n-1} and the volume V to r^n . Accordingly,

$$S \propto V^{1-1/d}. \tag{5.8}$$

The surface of a Bethe lattice grows exponentially with distance (in lattice units) from the origin:

$$S_\beta(r) = z(z - 1)^{r-1}. \tag{5.9}$$

Its volume is the sum (over r) of the surface shells:

$$V_\beta(r) = 1 + \sum_{i=1}^r z(z - 1)^{r-1} = \frac{z(z - 1)^r}{z - 2} - \frac{2}{z - 2}. \tag{5.10}$$

For all but small r ,

$$S_\beta(r) \simeq \frac{z - 1}{z - 2} V_\beta(r), \tag{5.11}$$

that is, $S_\beta(r) \propto V_\beta(r)$. From (5.8), this occurs for infinite dimension only.

It can also be shown, for a finite chain in dimension D , that the fraction of paths that it can take on which contains loops vanishes as $D \rightarrow \infty$. The Bethe lattice, by inspection, contains no loops; this feature is crucial to our ability to solve it.

5.3.2 PERCOLATION THRESHOLD

Consider an arbitrary site on an arbitrary cluster. If the cluster is infinite, at least one of the branches connected to the site must lead to infinity. At each step away from the site, we are faced with $z - 1$ new sites, $(z - 1)p$ of which are, on average, occupied. If this number is smaller than unity, the probability of finding an occupied path goes to zero exponentially with distance from the original site. We therefore will find an infinite cluster only if $(1 - z)p \geq 1$, that is,

$$p_c = \frac{1}{z - 1}. \tag{5.12}$$

5.3.3 INFINITE CLUSTER STRENGTH

Above p_c , within an infinite Bethe lattice there exists an infinite cluster. Clearly, as $p \rightarrow p_c$, the infinite cluster occupies an increasing fraction of the total number of sites in the lattice. Call this fraction the cluster strength P ; it is the probability that an arbitrary site belongs to the infinite network.

For convenience set $z = 3$ (Figure 5.6, left). We can calculate P as follows. Let Q be the probability that an arbitrary site is not connected to infinity through a particular branch emanating from that site. Now the probability that a site is not connected to an infinite network (1-P) can occur in two ways: either the site itself is not occupied (with probability $1 - p$) or none of its $z = 3$ branches connects to infinity (with probability Q^3). Accordingly,

$$1 - P = (1 - p) + pQ^3 \tag{5.13}$$

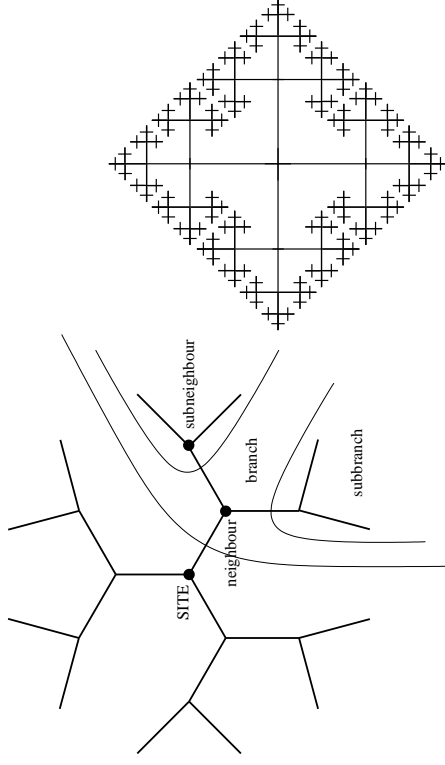


Figure 5.6: Left: A Bethe lattice with coordination number $z = 3$. Although the central site above appears to be the origin, an infinite Bethe lattice is translationally invariant. (The distorted metric is an artifact of embedding the lattice in 2-space.) Right: Another Bethe lattice with $z = 4$.

$$\Rightarrow P = p(1 - Q^3). \tag{5.14}$$

To solve for Q , we note that a branch can not reach infinity in one of two ways: either the neighbour itself is not occupied (with probability $1 - p$), or, if it is occupied, both its adjoining subbranches do not connect to infinity (with probability pQ^2). It follows that

$$Q = (1 - p) + pQ^2, \tag{5.15}$$

which has solutions

$$Q = 1 \quad \text{and} \quad Q = \frac{1 - p}{p}. \tag{5.16}$$

Substituting (5.16) into (5.14) yields

$$P = 0, \quad Q = 1, \tag{5.17}$$

corresponding to the case $p < p_c$, and

$$P = p - \frac{(1-p)^3}{p^2}, \quad Q = (1-p)/p, \quad (5.18)$$

for the case $p > p_c$.

5.3.4 MEAN CLUSTER SIZE S

As in one dimension, the mean cluster strength is the average size of the cluster to which an arbitrary occupied site belongs. Again we set $z = 3$. Let T be the average number of sites within a branch to which a site is connected. The mean cluster size may then be written

$$S = 1 + 3T. \quad (5.19)$$

We can solve for T by noting that, if a neighbour is unoccupied, $T = 0$; if it is occupied, T includes the two connecting subbranches in addition to the occupied site itself. Accordingly,

$$\begin{aligned} T &= (1-p)0 + p(2T+1) & (5.20) \\ \Rightarrow T &= \frac{p}{1-2p}. & (5.21) \end{aligned}$$

Substituting (5.21) into (5.19) yields

$$S = \frac{1+p}{1-2p}. \quad (5.22)$$

The mean cluster size S and the infinite cluster strength P are the order parameters which describe the system below and above threshold. How do they behave in the vicinity of p_c ? To be continued next time.

5.3.5 BEHAVIOUR NEAR p_c

Below p_c , (5.22) tells us that as the mean cluster size $S \rightarrow \infty$ as $p \rightarrow p_c$. Expanding S about p_c , we find the leading order behaviour to be

$$S \propto \frac{1}{p_c - p} \equiv (p_c - p)^{-\gamma}, \quad \gamma = 1. \quad (5.23)$$

Just above p_c , there exists a weak infinite cluster, whose leading order behaviour is (from (5.18))

$$P \propto p - p_c \equiv (p - p_c)^\beta, \quad \beta = 1. \quad (5.24)$$

Equations (5.23) and (5.24) are examples of scaling laws: near the critical point quantities of interest behave according to simple geometric relations. Scaling laws are observed in other (finite) dimensions, but there the exponents are not so simple; for instance, in two dimensions P has the same form near p_c but $\beta = \frac{5}{36}$. Remarkably, critical exponents do not depend on the lattice structure or the type of percolation (bond or site) but on dimension only. This is interesting, as it is not so easy for *humans* to give a rigorous definition to lattice dimension.

Equally surprising, for all $d \geq 6$ the various exponents are constant (independent of d) and identical to the Bethe lattice values derived above. The sixth dimension is the so-called upper critical dimension for percolation, with limiting values given by the Bethe lattice (or any other $d \geq 6$).

Other critical exponents will be introduced in later lectures.