EXAMPLES CLASS IN MATHEMATICAL PHYSICS

III. Waves and Fourier Transforms (20.1.99)

Commentary: Section A gives a brief recap on Fourier transforms. Fourier transforms and Fourier series (see sheet II) can both be thought of in terms of eigenfunction expansions. However, a Fourier transform rather than a Fourier series is needed whenever an infinite domain, rather than a finite one, is involved.

Section B shows the importance of Fourier transforms in dealing with wave equations. These become simpler when written in terms of Fourier components: the differential equation is converted into an algebraic one, conveniently written as a relation between frequency and wavenumber called the *dispersion relation.* This is illustrated by the case of waves on a string.

Section C applies the Fourier transform method to another important equation of classical physics: the *diffusion equation* for the spreading of molecules of a gas through a medium such as air. (This equation was derived formally in the Vectors sheet.) The same method allows a solution to a wide range of initial value problems in many areas of physics.

Section D is similar to C, but concerns the propagation of wave packets; it involves a study of group and phase velocities.

Don't be put off if you can't finish all the Sections in the time available; it is better to do part of the sheet properly than skim through the whole thing.

A. Fourier Transforms

A1. The **Fourier Transform** (FT) of a function $F(x)$ is defined as $f(k)$ where

$$f(k) = \int_{-\infty}^{\infty} F(x)\, e^{-ikx}\, dx\ . \tag{1}$$

Fourier's theorem then states that

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk. \tag{2}$$

Prove that in general

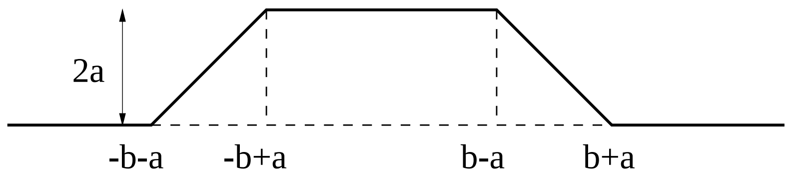
(i) The FT of $\frac{dF(x)}{dx}$ is $ikf(k)$;

(ii) The FT of $F(x-b)$, with b a constant, is $e^{-ibk}\, f(k)$.

Calculate the Fourier Transforms of the following functions:

(iii) a top hat function of height λ and width $2a$, centred on the origin.

(iv) as shown below:



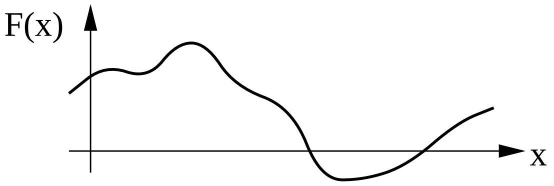
[Hint: first calculate dF/dx and then apply the rules given above. Answer: $f(k) = 4\sin(ka)\sin(kb)/k^2$.]

B. Wave Equations and Dispersion Relations

Wave problems in an infinite medium are most easily addressed by Fourier Transform methods. For example, consider displacements $F(x,t)$ of an infinitely long string, as shown on the next page. These are governed by the Wave Equation

$$\rho \frac{\partial^2 F}{\partial t^2} = T \frac{\partial^2 F}{\partial x^2}$$

where T the tension in the string, and ρ the mass per unit length.



(i) Show, using the theorem in A1(i) above, that if

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k,t)\, e^{ikx}\, dk$$

then the fourier components obey

$$\frac{\partial^2 f(k,t)}{\partial t^2} = -(T/\rho)k^2 f(k,t)\ .$$

Hence show that a solution is $f(k,t) = f(k,0)\, e^{-i\omega(k)t}$, and give an equation for the *dispersion relation* $\omega(k)$. What is the wave speed?

(ii) Find the dispersion relation $\omega(k)$ for the following equation, which describes the motion of a vibrating string in the presence of air friction γ per unit length:

$$\rho\frac{\partial^2 F}{\partial t^2} + \gamma\frac{\partial F}{\partial t} = T\frac{\partial^2 F}{\partial x^2}\ .$$

Give a limiting form of your expression for the case $\gamma^2 \ll Tk^2\rho$, and show that in this limit the amplitude of a wave falls by a factor e in the course of N cycles where $N = \frac{(T\rho)^{1/2}k}{\pi\gamma}$.

C. Uses of Fourier Transforms

In sheet II we solved various initial value problems in terms of Fourier *series*. These are appropriate when the disturbance is, for all times, limited to a finite interval. Each *discrete* Fourier mode had a simple time evolution; by finding the mode amplitudes at time zero and evolving these separately, the displacement or temperature profile at arbitrary times t could be synthesized.

Here we are interested in disturbances on an infinite interval. Instead of a Fourier series (sum of discrete modes) we have an integral over modes, or the Fourier transform. Otherwise the principles are the same. We illustrate them here with an example from diffusion, and in Section D, one from wave motion.

C1. A problem of diffusion. "A physics laboratory contains a corridor, of unbounded length, with fire-doors located at positions $x = a$, $x = -a$. Overnight, while the doors are closed, a small amount of highly noxious gas, of diffusivity D, is released into the space between the doors, building up to a uniform concentration C_o. In the morning the doors are suddenly opened. Give an expression for the concentration profile $C(x,t)$ as a function of the elapsed time t."

(i) Our first task is to find an exact formal solution for $C(x,t)$ as a function of time. The equation for diffusion of a gas is

$$\frac{\partial C}{\partial t} = D\nabla^2 C \tag{3}$$

[See the vectors sheet for a discussion and derivation of this equation.] What is the initial condition?

To find a formal solution, we can decompose $C(x,t)$ into its Fourier components:

$$C(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} f(k,t)$$

Since Eq.(3) is linear in C, each such component will obey it separately. Hence show that $f(k,t)$ can be written as $f(k,0)e^{-i\omega(k)t}$ where the frequency $\omega(k)$ depends on k, and find the appropriate "dispersion relation" $\omega(k)$ in this problem. [With hindsight, it would have been more convenient to write the "phase factor" as $e^{-\mu(k)t}$, with μ real, but the end result is of course the same.]

(ii) The amplitudes $f(k,0)$ can be found from the initial conditions, by exploiting the orthogonality relation for the functions e^{ikx}:

$$\int_{-\infty}^{\infty} dx\, e^{ik'x}\, e^{-ikx} = 2\pi\delta(k-k')$$

where δ is the Dirac delta function.

Prove using this relation that

$$f(k,0) = \int_{-\infty}^{\infty} dx\, e^{-ikx} C(x,0)$$

and calculate the form of $f(k,0)$ explicitly for the present problem. (This method is, of course, equivalent to using Fourier's theorem (2).) Hence construct a formal solution for the concentration profile $C(x,t)$ at time t.

Answer:

$$C(x,t) = \frac{C_o}{2\pi}\int_{-\infty}^{\infty} dk\, \left[\frac{2\sin(ka)}{k}\right]\, e^{ikx-Dk^2t} \tag{4}$$

C2. Approximating the Solution. The formal result just given is complete, but hard to use. Suppose we only need to know how long it is before the concentration of gas in the corridor reaches some low (safe) level. To find this we will need a simplified form of our expression, valid for the limit of long times. The procedure illustrates two important points: (a) the use of physical arguments to select a good mathematical approximation; and (b) evaluation of integrals involving the gaussian distribution by completing the square.

(i) Sketch the behaviour of $\frac{\sin ka}{ka}$ as a function of ka. Considering that e^{-Dtk^2} decays rapidly at large enough k, give a criterion (in terms of t) under which Eq.(4) can be written as

$$C(x,t) \simeq \frac{C_o a}{\pi}\int_{-\infty}^{\infty} dk\, e^{ikx-Dk^2t} \tag{5}$$

Interpret your criterion in physical terms.

(ii) Consider the following integral

$$I = \int_{-\infty}^{\infty} dk\, e^{-bk^2} = \sqrt{\frac{\pi}{b}}$$

[If this is unfamiliar, prove it by evaluating I^2 in circular polar coordinates.] With the help of this result, prove the relation:

$$\int_{-\infty}^{\infty} dk\, e^{-bk^2+ck} = \sqrt{\frac{\pi}{b}}\, e^{c^2/4b}$$

Hence obtain an asymptotic formula for $C(x,t)$ at long times.

$$\text{[Answer:} \quad C(x,t) = C_0\sqrt{\frac{a^2}{\pi D t}}\, e^{-x^2/4Dt}\,.\text{]}$$

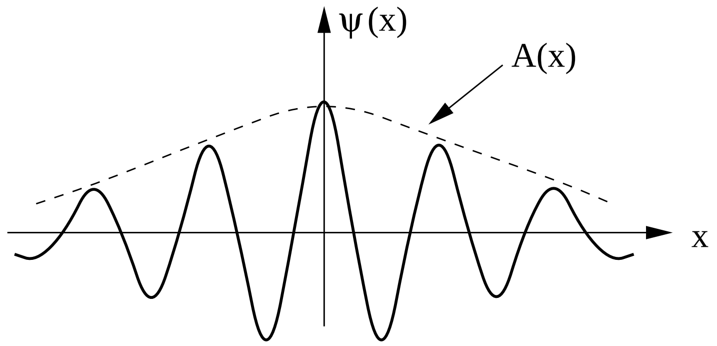
Comment on the effect of the initial conditions: what would would have happened if the initial profile had been some other localised shape with width of order a?

(iii) Suppose that $a = 2$m, $C_0 = 1$ppm and the noxious gas is hydrogen selenide ($D = 10^{-4}\mathrm{m}^2\mathrm{s}^{-1}$). If diffusion were the only process at work in dispersing the gas, how long would it be before the maximum concentration in the corridor falls to a safe level of 0.05 ppm? [Ans: approximately 2 months.] What other processes are likely to matter in practice?

D. Propagation of a Wavepacket with Dispersion.

We now turn from a diffusion to a propagation problem. Frequently we are interested in the propagation of a pulse or wave packet rather than a wave of uniform amplitude in space. Examples include the emission of photons by atoms, signals traveling down a transmission line, the representation of particles by waves in quantum mechanics, and the propagation of deep water waves and ripples. As in the previous problem, the pulse must be synthesised from waves of different wavevectors. When a wave propagates through a *dispersive* medium, for which $c = \omega(k)/k$ is not a constant, different Fourier components travel at different velocities, and hence the pulse shape is distorted during propagation. For example, particle waves, even in the absence of a medium, are intrinsically dispersive ($\omega = \hbar k^2/2m$), as are deep water waves ($\omega^2 = gk$) .

D1. Consider a general one-dimensional problem in which we set up a *wave packet* at time $t = 0$, which we will assume to be symmetrical about the origin. This defines the initial condition of the problem: we must evolve these forward in time to find the disturbance $\psi(x,t)$. The evolution is governed by an arbitrary dispersion relation $\omega(k)$. The "carrier" wave, that is the basic wave underlying the initial pulse, has wavenumber q and it is assumed that the envelope function of the pulse $A(x)$ is much broader than one wavelength (see figure):



The initial profile is thus

$$\psi(x,0) = A(x)e^{iqx} .$$

Show that the total disturbance can be written initially as

$$\psi(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i(k+q)x} B(k)$$

where

$$B(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} A(x) .$$

This corresponds to a distribution of fourier components $B(k)e^{i(k+q)x}$ centered (in wavenumber space) around the carrier wave e^{iqx}.

(i) Assume that each Fourier component evolves in time according to some arbitrary dispersion relation $\omega(q)$. Show that the disturbance at time t can be written as:

$$\psi(x,t) = e^{i(qx-\omega(q)t)} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} B(k)\, e^{ik(x-\omega'(q)t)} \times e^{-i\frac{1}{2}\omega''(q)k^2t+\dots}\, dk \tag{6}$$

where $\omega'(q) \equiv \frac{d\omega}{dq}$ and $\omega''(q) \equiv \frac{d^2\omega}{dq^2}$. By considering separately each of the three exponential factors, interpret this formula physically and derive expressions for (a) the *phase velocity* of a particular crest; (b) the *group velocity* of the wave train; and (c) the time it takes for the train to spread or alter its shape appreciably. [Note that for times much longer than this, the neglected terms (+...) in the third exponential may become important and the treatment starts to break down. However some important cases (e.g. quantum particle waves) have a quadratic dispersion relation and there are no third and higher order terms; Eq.(6) is then exact.]

(ii) A general maxim is "wide in x-space, narrow in k-space". To demonstrate this, calculate $B(k)$ explicitly for the case of a gaussian wave packet $A(x) = e^{-x^2/4\lambda^2}$ with width $\lambda \gg \pi/q$. Hence show that the wave's components at wavenumber $k+q$ only have significant weight if $k < 1/\lambda \ll q$. [Hint: recall the method of completing the square in C2 (ii).] This justifies our description of the wave packet as having a well-defined carrier frequency, as well as the Taylor expansion of ω on which Eq.(6) depends.