EXAMPLES CLASS IN MATHEMATICAL PHYSICS

I. Vectors and Vector Fields (18.11.98)

Commentary: This example sheet is in three parts. Some of the material should already be familiar, but the way it is presented may differ from lectures. An understanding of the properties of vectors and vector fields is essential to the rest of the physics course.

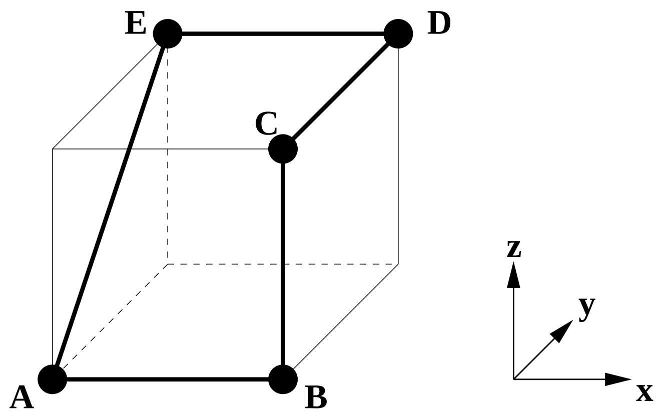
Section A: Some drill question on vectors to remind you how they can simplify problems that are cumbersome to do by other means. This includes a treatment of the concept of vector area.

Section B: Bookwork on vector fields: the divergence theorem and Stokes theorem. Make sure you understand this section thoroughly.

Section C: Use of vector algebra to derive physical laws from elementary principles or postulates. This theme will recur in a later sheet on electromagnetism: here we go through the simpler case of deriving the diffusion equation.

A. Examples on use of vectors

A1. Consider a rigid wire loop $ABCDEA$ inscribed on a unit cube, with equal point masses at the positions indicated in the figure below:



Calculate the vector from A to the centre of mass. Hence show that, if the loop is suspended under gravity from the point A, the vector AD makes an angle $\theta = \cos^{-1}\left(\sqrt{32/33}\right)$ to the vertical. [Hint: use the dot product.]

A2. In the presence of a magnetic field $\mathbf{B}(\mathbf{r})$, the magnetic flux through an arbitrary closed curve may be expressed as

$$\Phi = \int_{\Sigma} \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S} \qquad (1)$$

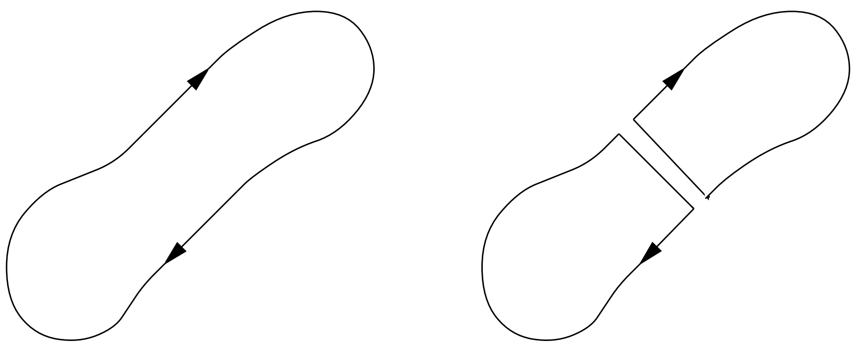
where Σ is *any* closed surface bounded by the loop. (If you are not sure why this is true, pay special attention to B5 below.) The element of surface area $d\mathbf{S}$ is normal to the surface and its sense defined by the right hand screw convention.

In the case where $\mathbf{B}(\mathbf{r})$ is a constant vector $\mathbf{B}$, Eq.1 can be written

$$\Phi = \mathbf{B} \cdot \mathbf{S} \qquad (2)$$

where $\mathbf{S} = \int_{\Sigma} d\mathbf{S}$ is called the *vector area* of the loop. Note that $\mathbf{S}$ must indeed be a vector, since its dot product with an arbitary vector $\mathbf{B}$ yields a scalar, Φ.

(a) Show that if a large loop is decomposed into two smaller loops in the following manner, the vector areas obey $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$:

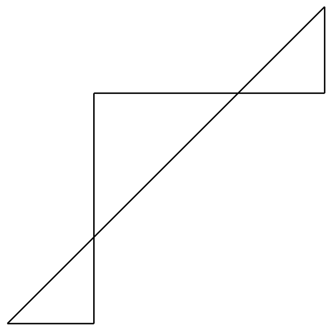


(b) Calculate $\mathbf{S}$ for the loop $ABCDEA$ in question A1 above.

(c) Hence show that the flux threaded by the loop in a constant field of magnitude B along the AD direction is $\Phi = -\frac{B}{2\sqrt{3}}$.

(d) In fact the vector area of an arbitrary loop has the following physical interpretation: for any direction specified by a unit vector $\mathbf{u}$, the quantity $\mathbf{S}\cdot\mathbf{u}$ is the *area of the projection of the loop onto a plane perpendicular to* $\mathbf{u}$. Prove this, *e.g.*, by considering such a loop in a field $\mathbf{B} = B\mathbf{u}$ and deciding what deformations can be made to the curve without changing the total flux enclosed.

(e) Along the direction $\mathbf{u} = \frac{3}{2\sqrt{2}}(1, -\frac{1}{4}, \frac{1}{4})$, the projection of the loop $ABCDEA$ looks like this:



Show that $\mathbf{S}\cdot\mathbf{u} = 0$: how is this reconciled with the above sketch?

A3. The base vectors of a certain crystal structure are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (not necessarily the same length nor at right angles). A particular plane of atoms intersects the base vectors at the points $\mathbf{a}/h, \mathbf{b}/k, \mathbf{c}/l$.

(a) Show that a normal to the plane of atoms is the vector $\mathbf{N} = h\mathbf{a}^* + k\mathbf{b}^* + l\mathbf{c}^*$, where $\mathbf{a}^*$, *etc.* are the reciprocal vectors (note that $\mathbf{N}$ is not a unit vector).

(b) Show that the perpendicular distance between consecutive planes of atoms intersecting the base vectors at $(\mathbf{a}/h, \mathbf{b}/k, \mathbf{c}/l)$; $(2\mathbf{a}/h, 2\mathbf{b}/k, 2\mathbf{c}/l)$; *etc.* is $1/|\mathbf{N}|$.

[*Reminder*: Reciprocal vectors are defined as $\mathbf{a}^* = (\mathbf{b}\times\mathbf{c})/[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]$, *etc.* . Points $\mathbf{r}$ in a plane with unit normal $\mathbf{n}$ and a perpendicular distance d from the origin obey $\mathbf{n}\cdot\mathbf{r} = d$.]

B. Basic results for handling vector fields

B1. Consider a particle moving along a line in a potential $U(x)$, which incurs a gain in kinetic energy $U(x_1) - U(x_2)$ in going from x_1 to x_2. Of course this is the same as the work done as found by integrating the force:

$$U(x_1) - U(x_2) = \int_{x_1}^{x_2} \left(-\frac{dU(x)}{dx}\right) dx \qquad (3)$$

This equality relates two different formulations of the same physical law; in this case the law is that energy is conserved, not only for finite changes, but for differential ones. More generally one can relate the value of a physical quantity on the boundary of some region with its derivatives in the interior.

Such relations are important for laws that are formulated in terms of vector fields, such as electric and magnetic fields. Two fundamental results are, for any vector field $\mathbf{F}(\mathbf{r})$,

1. Divergence theorem:

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_V \operatorname{div}\mathbf{F}\, dV \qquad (4)$$

where $d\mathbf{S}$ is the element of a closed surface S enclosing a volume V. Note that $d\mathbf{S}$ is taken to be directed *normally outwards* to the increment of surface area. The integral on the right hand side is over the volume bounded by V. You should of course know that $\operatorname{div}\mathbf{F} \equiv \nabla \cdot \mathbf{F}$. Hence div and ∇ are *vector operators* although, by convention, they are not usually written in bold type.

2. Stokes theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} \tag{5}$$

where C is a closed curve and S is any open surface bounded by the loop C. The vector operator curl is defined by $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$. If $\mathbf{F}$ is a field of force, $\oint \mathbf{F} \cdot d\mathbf{l}$ is *minus* the work needed to take a particle once round the loop.

B2. Use the product rule to show that

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)$$

Note that the left hand side is often written as $\nabla \cdot \phi \, \nabla \psi$. The brackets are technically redundant since $\nabla\cdot$ must act on a vector which means that alternatives such as $(\nabla \cdot \phi)(\nabla \psi)$ are not legitimate expressions. But if in doubt, always use extra brackets to make things clear. [Even better: use the suffix notation – see later sheets.]

By considering $\mathbf{F} = \phi \nabla \psi$, derive *Green's theorem*

$$\int_V \left[\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)\right] dV = \int_S \phi \nabla \psi \cdot d\mathbf{S} \tag{6}$$

B3. Show that if $\mathbf{n}$ is a constant vector and U a scalar function of position, $\operatorname{div}(U\mathbf{n}) = \mathbf{n} \cdot \nabla U$. Hence show from the divergence theorem that

$$\int_S U \, d\mathbf{S} = \int_V (\nabla U) \, dV \tag{7}$$

Notice the very close resemblance to Eq.3 above, which is just the one dimensional version of this general result. By a similar treatment of curl $(U\mathbf{n})$, show that

$$\oint_C U d\mathbf{l} = \int_S (d\mathbf{S} \times \nabla U) \tag{8}$$

B4. A vector field $\mathbf{F}$ for which $\mathrm{curl}\,\mathbf{F} = 0$ is called *irrotational* or *conservative*. Show that each of the following four definitions of a conservative field implies the other three:

(a) $\mathbf{F} = -\nabla\phi$, where ϕ is a scalar function of position. [This ϕ is called the scalar potential of the vector field $\mathbf{F}$.]

(b) $\mathrm{curl}\,\mathbf{F} = 0$.

(c) $\oint \mathbf{F}.d\mathbf{l} = 0$ for any closed loop.

(d) $\int_a^b \mathbf{F}.d\mathbf{l}$ is independent of the path from a to b.

Note that these criteria are *not* all equivalent for a region that is not simply connected. (A simply connected region is one in which every closed curve can be continuously shrunk to a point.) Consider a region R consisting of all space exterior to a tyre. If $\mathrm{curl}\,\mathbf{F}$ vanishes everywhere outside the tyre, but is nonzero inside, which of the four statements given above still hold in R?

B5. A vector field $\mathbf{F}$ for which $\mathrm{div}\,\mathbf{F} = 0$ is called a solenoidal field. If $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$ then $\mathrm{div}\,\mathbf{B} = 0$. Conversely, any solenoidal field $\mathbf{B}$ can be written as the curl of some vector field $\mathbf{A}$. In magnetism $\mathbf{A}$ is called the vector potential. Note that $\mathbf{A}$ is not unique: the transformation $\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + \nabla\phi(\mathbf{r})$, with ϕ an arbitrary scalar quantity, does not change the resulting magnetic field $\mathbf{B}$ (why?).

Derive the following useful result for the curl of the curl of an arbitrary vector field:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$

[*Hint:* write it out explicitly for one cartesian component.] Make sure you know exactly what $\nabla^2\mathbf{F}$ means when $\mathbf{F}$ is a vector field. Find curl $\mathbf{B}$ for the case when the magnetic vector potential obeys $\mathbf{A}(x, y, z) = (yx, 0, xz)$.

C. The diffusion equation

C1. Suppose we have a system of diffusing particles in free space whose density at a point $\mathbf{r}$ is $\rho(\mathbf{r})$, so that the number of particles in some volume V is $\int \rho dV$. We assume the particles to be so numerous that ρ can be treated as a continuous quantity; clearly it is a scalar.

The *particle current density* $\mathbf{J}$ is defined so that $dn = \mathbf{J} \cdot d\mathbf{S}\, dt$ is the net number of particles crossing a surface element $d\mathbf{S}$ in a short time dt. (Note that $\mathbf{J}$ must be a vector since its dot product with the vector $d\mathbf{S}$ yields a scalar quantity.) Each particle moves randomly, and so when the density is uniform the number of particles crossing the surface element from one direction is cancelled by those moving the opposite way; however when ρ is nonuniform, a nonzero particle current should be set up. This obeys

$$\mathbf{J} = -D\nabla\rho \tag{9}$$

where D is a constant called the diffusivity.

You should be able to derive this form, for example from the kinetic theory of gases, but another way to arrive Eq.9 is as follows. The right hand side must be a vector that depends linearly on ρ and its gradients. The quantity $\nabla\rho$ is the simplest such vector that can be constructed, although there are in principle others such as $\nabla(\nabla^2\rho)$ involving higher powers of the ∇ operator. The neglect of higher powers of ∇ should be a good approximation when ρ is slowly varying with position. [In fact a close examination of the kinetic theory confirms that these higher terms are indeed present.]

C2. Show that the continuity equation expressing conservation of particles is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{10}$$

and hence derive the diffusion equation

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho \tag{11}$$

Solution methods for this and similar equations will be examined on later example sheets.

We normally expect the diffusivity D to be independent of position, as assumed above, but this is not necessarily the case (e.g. for vacancies diffusing through a solid whose properties vary from place to place). Show that in this case the proper generalization of Eq.(11) is

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho + \nabla D \cdot \nabla \rho \,. \tag{12}$$

C3. By an almost identical series of mathematical steps, one can use the conservation of energy to derive the **heat diffusion equation**

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

for the temperature T in a material of uniform thermal conductivity K, heat capacity C and density ρ; the parameter $\kappa = K/C\rho$. If you have any doubt about how this is done, try working through the procedure with the previous example as a pattern.

C4. The same mathematical steps do not work for *charged* particles such as electrons in a metal. A charge distribution ρ is set up in a uniform medium of constant conductivity (inverse resistivity) σ. Show that instead of Eq.11, conservation of charge carriers leads now to the relation:

$$-\frac{\partial \rho}{\partial t} = \frac{\sigma \rho}{\epsilon_r \epsilon_0}$$

In this case, where does the general mathematical argument leading to the diffusion equation break down? [Hint: recall from electric theory that the current is now $J = \sigma \mathbf{E}$.]