EXAMPLES CLASS IN MATHEMATICAL PHYSICS

VII. Variational Calculus in Physics (17.2.99)

Commentary: The calculus of variations involves finding the stationary value of a quantity with respect to variations in a function upon which it depends. This involves the idea of a *functional*, which is introduced in Section A. There we also recall one of the main result of variational calculus, namely Euler's equation. Section B contains some straightforward examples from variational calculus: you can miss this section if you are familiar with the material from Maths courses. In Physics the variational method takes a deeper significance. Most of the differential equations of Physics can be framed as *variational principles*: in each case there is a functional that when made stationary yields the required equation, for instance the equations of mechanics, quantum mechanics and optics. In this sheet, we consider only one-dimensional problems but the correspondence applies also in higher dimensions.

In Section C we derive the Lagrangian formulation of mechanics from Hamilton's principle, and apply the methods to a problem involving a bead on a rotating hoop. This problem can be treated rapidly using the Lagrangian method, although of course the results can be also obtained (much more painfully) directly from Newton's laws. Section D covers the method of undetermined multipliers. This is an important way to deal with problems that involve minimisation of a function subject to constraints. We take the example of the mathematical shepherd.

A. Introduction

A1. In mathematics courses you will have met the idea of a *functional*: a quantity, say I, the value of which depends of the choice of *function*, say $y(x)$. An example is

$$I[y(x)] = \int_a^b F\left(y(x), y'(x), x\right)\, dx \tag{1}$$

where F is a known function of y, $y'(\equiv dy/dx)$ and x (the case $F(y, y', x) = Ay'^2 + By^2$ is a frequently encountered example).

The difference between a *function* such as F, and a *functional* such as I, is that F depends locally on the values of y and its derivatives at each particular place (each x), whereas I depends on the *whole shape* of $y(x)$ for x in the range $a \le x \le b$. The arguments of the function F are placed in the normal round brackets ( ). To indicate that I is a *functional* square brackets [ ] are often used.

Which of the following is a function, and which a functional, of $y(x)$?

(i) $\sin y(x)$

(ii) $\int_{-\infty}^{\infty} y(x) e^{ikx}\, dx$ (Fourier transform)

(iii) The maximum value of $y(x)$ in some interval

(iv) $\int_0^{\infty} e^{-\omega y(x)^2}\, d\omega$

A2. Varying the choice of function $y(x)$ causes I in Eq.(1) to take different values. Show that under an arbitrary *small* change $y(x) \to y(x) + \eta(x)$, where $\eta(a) = \eta(b) = 0$, $I[y(x)] \to I + \delta I$ where

$$\delta I[y(x), \eta(x)] = \int_a^b \left[\left(\frac{\partial F}{\partial y} \right) - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] \eta(x)\, dx\ . \tag{2}$$

[You will need to integrate by parts.] The condition that the path $y(x)$ must obey, for I to be stationary, is then expressed as

$$\delta I[y(x), \eta(x)] = 0 \qquad \forall\ \eta(x) \tag{3}$$

(often written as $\delta I = 0$) which, since $\eta(x)$ is arbitrary, implies *Euler's Equation*

$$\left(\frac{\partial F}{\partial y}\right)_{y',x} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)_{y,x} = 0. \tag{4}$$

Note that in this context

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{dy'}{dx}\frac{\partial}{\partial y'} \tag{5}$$

One of the biggest problems students have in dealing with Euler's equation is to remember exactly which variables are being held constant during partial differentiation. It may be wise to explicitly write these as suffices (as in (4)) until you are confident with the method.

(i) Show that if $F = F(y', x)$ does not depend on the variable y, then

$$\frac{\partial F}{\partial y'} = \text{const.} \tag{6}$$

(ii) Show that if $F = F(y, y')$ does not explicitly depend on x then

$$y'\frac{\partial F}{\partial y'} - F = \text{const.} \tag{7}$$

[Hint: d/dx of L.H.S. must be zero. Use (5) and (4).]

B. Simple examples of variational calculus

(Omit this section if these examples are familiar from Maths courses)

B1. Prove that the shortest distance between two points is a straight line.

[Hint: Remember that the length ds of an element of a curve $y(x)$ in the $x - y$ plane is $ds = \sqrt{(dx)^2 + (dy)^2} \equiv (1 + y'^2)^{1/2}dx$. Eq.(7) may be useful.]

B2. The Brachistochrone: A particle falls from rest down a smooth curved tube. The brachistochrone is the form the tube should have in order that the particle may arrive at a second fixed point in the least possible time. Derive the function $F(y, y')$ needed in in Eq.(1) for the functional I to correspond to the time of flight. [See hint for B1. Consider the particle's speed ds/dt when it has fallen to a level y below its starting point.]

Show that the required path $y(x)$ obeys $y(1+y'^2) = \text{constant}$. [This is satisfied by a *cycloid*, i.e., a curve given parametrically by

$$y = a(1-\cos 2\psi)$$

$$x = a(2\psi - \sin 2\psi)$$

where ψ is the angle the path makes with the vertical. If you want to check, use $\partial y/\partial x = (\partial y/\partial \psi)(\partial \psi/\partial x)$.]

C. Analytical Dynamics

C1. The laws of mechanics can be formulated either as *Newton's Laws of Motion* or in terms of *Hamilton's principle*, which postulates that

$$\delta \int_{t1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0 \tag{8}$$

under variations in $q_i(t)$. Here q_i, $i = 1 \ldots n$ are a set of coordinates which must must be chosen so that they are linearly independent and equal in number to the degrees of freedom of the system. L is called the *Lagrangian function* and is constructed as $L = T - U$ where T is the total kinetic energy and U the total potential energy of the system.

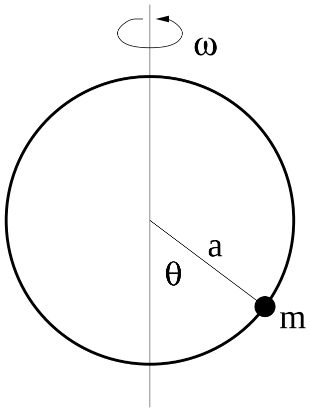
(i) Consider a system of one degree of freedom q. Show that Hamilton's principle implies

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{9}$$

[For a system with many degrees of freedom q_i, this equation holds for each q_i separately, since Hamilton's principle includes variation of any one of the q_i's with the others held fixed. The whole set of equations together are known as *Lagrange's equations*.]

(ii) Show that for a particle moving in one dimension, equations (6) and (7) correspond to conservation of linear momentum and energy respectively. Comment on the conditions for their being obeyed.

C2. Consider a particle of mass m that can slide freely on a circular hoop of radius a, which rotates about a vertical axis at fixed angular velocity ω:



(i) Show that the Lagrangian of the bead is

$$L(\theta, \dot{\theta}) = \frac{1}{2} m a^2 \left(\omega^2 \sin^2 \theta + \dot{\theta}^2 \right) + mga \cos \theta \tag{10}$$

(ii) Hence obtain the bead's equation of motion in the form

$$\ddot{\theta} = \left(\omega^2 \cos \theta - g/a \right) \sin \theta \tag{11}$$

(iii) Use this to show that, for $\omega^2 a < g$, the particle can remain in stable equilibrium at the bottom of the hoop, whereas for $\omega^2 a > g$ this position is unstable. Find the new position θ^* of stable equilibrium, and calculate the angular frequency Ω of small oscillations about this position. [Answers: $\theta^* = \cos^{-1}(g/a\omega^2)$; $\Omega^2 = \omega^2 - (g/a\omega)^2$.]

D. Method of Undetermined Multipliers

Suppose we now want to find the path $y(x)$ that makes a functional $I[y(x)]$ (Eq.1) stationary subject to a *constraint*, for instance that a second functional,

$$J[y(x)] = \int_a^b G(y, y', x)\, dx$$

is held constant. The constraint restricts the allowed types of variation $\eta(x)$ and hence the criterion for stationarity of I differs from Eq.(3). A *restricted* set of variations $\bar{\eta}(x)$ about a given path $y(x)$ may be defined as those that obey

$$\delta J[y(x), \bar{\eta}(x)] = 0 \ .$$

These variations are those that satisfy the constraint; hence the condition on $y(x)$ that I is stationary, subject to the constraint of constant J, may be written

$$\delta I[y(x), \bar{\eta}(x)] = 0 \qquad \forall\, \bar{\eta}(x) \tag{12}$$

D1 Show that the following condition on $y(x)$ is sufficient to ensure (12) holds:

$$\delta I[y(x), \eta(x)] = \lambda\, \delta J[y(x), \eta(x)] \qquad \forall\, \eta(x) \tag{13}$$

where $\eta(x)$ is now an *unconstrained* variation and λ is an arbitrary constant.

Clearly, Eq.(13) is equivalent to seeking stationary values of a modified functional $K = I - \lambda J$, where the problem is to be solved now with *no* constraint on the type of variation allowed. The undetermined multiplier λ is treated as a parameter, which can later be fixed from considering boundary conditions, etc..

D2. The Mathematical Shepherd:

A straight drystone wall includes two posts, a distance a apart. A shepherd has a length b (with $\pi a > 2b > 2a$) of portable fencing, one end of which he can attach to each post. He wants to enclose the maximum grazing area for his sheep between the fence and wall.

(i) Show that, to maximise the area, the locus of the fence $y(x)$ should be chosen to obey

$$\lambda(1 + y'^2)^{-1/2} - y = c$$

where λ and c are constants.

[Hint: You should find that F in Eq.(1) has no explicit x dependence. This allows the conservation equation (7) to be used.]

(ii) Verify that this represents an arc of a circle.

[Note: λ and c are in principle fixed by the values of b and a. But it is easier to construct the appropriate arc by inspection, once the general solution is found.]