EXAMPLES CLASS IN MATHEMATICAL PHYSICS

VIII. Further Applications of Tensor and Vector Calculus (24.2.99)

Commentary: This is the second sheet on tensors. It follows on from Sheet 5. Section D of that sheet (on uses of the antisymmetric 3-tensor, ϵ_{ijk}) is reprinted here as an appendix, and you should complete it before starting the main part of this sheet, if you didn't do so last time.

As on sheet 5 we use underlining and bold notation interchangeably. Suffix notation and the summation convention are also used. Carets ($\hat{}$) denote unit vectors.

In Sections A and B tensor ideas are used to work through some results in rigid body dynamics. You should have covered this material in IB dynamics courses, and the formal development given here is good practice and will help deepen your understanding of moments of inertia, etc., as well as of tensors.

Section C introduces the concepts of *stress* and *strain* tensors in a solid. The linear relation between these is governed by a *fourth rank tensor*, whose properties are investigated.

Section D contains problems on the transformation properties of tensors: in mathematical courses, you may have seen tensors *defined* in terms of these transformation properties.

A. Rigid Body Rotations

A rigid body rotates with angular velocity $\underline{\omega}$ about an axis through its centre of mass at the origin, 0. The vector $\underline{\omega}$ points along the axis of rotation and determines the velocity of rotation at $\mathbf{r}$ according to $\mathbf{v} = \underline{\omega} \wedge \mathbf{r}$. [Make sure you can derive this if you are not sure where it comes from.]

(i) Consider the body as a collection of point masses $\delta m_{\mathbf{r}}$ with position vectors $\mathbf{r}$ relative to the centre of mass 0.

The tensor $\mathbf{I}$ is called the *inertia tensor* of the body. It is given by:

$$\mathbf{I} = \sum_{\mathbf{r}} r^2 \left(\underline{\underline{1}} - \hat{\mathbf{r}}\,\hat{\mathbf{r}}\right) \delta m_{\mathbf{r}} \tag{1}$$

Show, using Eq.(1), that the angular momentum $\mathbf{L}$ and the kinetic energy T of the body are:

$$\mathbf{L} = \mathbf{I} \cdot \underline{\omega} \tag{2}$$

$$T = \frac{1}{2}\underline{\omega} \cdot \mathbf{I} \cdot \underline{\omega} \equiv \frac{1}{2}\omega_i\, I_{ij}\, \omega_j \tag{3}$$

[You will need to expand the vector triple product $\mathbf{f} \wedge (\mathbf{g} \wedge \mathbf{h})$ – see Appendix.]

Thus $\mathbf{I}$ specifies the linear dependence of angular momentum L_i on the angular velocity ω_j (via $L_i = I_{ij}\,\omega_j$, Eq.(2)) and replaces the single number, the *moment of inertia*, that would be enough for a very symmetrical body, such as a sphere.

(ii) Find the components of I_{ij} in a Cartesian system, that is the usual basis set $\hat{\underline{x}}$, $\hat{\underline{y}}$, $\hat{\underline{z}}$. [Answer: $I_{xx} = \sum_{\mathbf{r}}(y^2 + z^2)\delta m_{\mathbf{r}}$, $I_{xy} = -\sum_{\mathbf{r}}(xy)\delta m_{\mathbf{r}}$, *etc.*.]

(iii) I_{ij} is a *symmetric tensor*, that is $\mathbf{I}.\mathbf{v} = \mathbf{v}.\mathbf{I}$ for any vector $\mathbf{v}$, which should be clear from the definition (1). Deduce it also from the matrix of components of I_{ij} found in (ii).

(iv) Take the continuum limit in Eq.(1) to obtain formally, as an integral, the inertia tensor of a body of variable density $\rho(\mathbf{r})$.

(v) Write down the inertia tensor of a body consisting of eight masses M at the corners of a light cubic frame of side a. Choose the coordinate system most convenient to you. What is the moment of inertia of this cube? Why can we talk of *the* moment of inertia when we strictly speaking have a tensor?

[In fact, even for an asymmetric body one sometimes speaks of *the* moment of inertia about some specific axis, $\hat{\mathbf{n}}$, say. This is written as $\hat{\mathbf{n}}.\mathbf{I}.\hat{\mathbf{n}}$ and gives the ratio of the $\hat{\mathbf{n}}$ component of angular momentum ($\mathbf{L}.\hat{\mathbf{n}}$) to the $\hat{\mathbf{n}}$ component of angular velocity ($\underline{\omega}.\hat{\mathbf{n}}$). Since in general $\mathbf{L}$ has components which are not parallel to $\hat{\mathbf{n}}$, this does not offer a complete description.]

(vi) **Angular velocity:** A body rotates about an axis through its centre of mass with an angular velocity $\underline{\omega}$. The velocity of a point in the body is $\mathbf{v} = \underline{\omega} \wedge \mathbf{r}$. Show that one can equally well express it as $\mathbf{v} = \underline{\underline{\omega}} \cdot \mathbf{r}$ where $\underline{\underline{\omega}}$ is an antisymmetric tensor with components:

$$\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{4}$$

(vii) Show that any vector $\mathbf{W}$, constant with respect to axes fixed in the body, changes with time in a space-fixed frame, by an amount $d\mathbf{W}$ in time dt:

$$d\mathbf{W} = (\underline{\omega} dt) \wedge \mathbf{W}$$

and hence that if $\mathbf{W}$ is not body-fixed but has a rate of change $(d\mathbf{W}/dt)_{body}$, the vector $\mathbf{W}$ in the space-fixed frame obeys

$$(d\mathbf{W}/dt)_{space} = (d\mathbf{W}/dt)_{body} + \underline{\omega} \wedge \mathbf{W} \ . \tag{5}$$

B. Principal Axes and Precession

We now use the above results to solve simply a difficult problem, the rotational motion of an asymmetric body.

(i) What is the inertia tensor of four point masses M fixed at the corners of a massless square frame of side a? What are the principal axes, and principal moments of inertia? Why are two of the axes indeterminate?

This is an example of a nontrivial body (unlike the sphere or cube) where the principal moments of inertia are not all the same. Now $\mathbf{L}$ need not be parallel to $\underline{\omega}$; for a body whose angular momentum is conserved the result is *free precession* of the angular velocity (see (iii) below).

(ii) It is clear from (1) that in general $\mathbf{I}$ for a body is constant in body-fixed axes, and so for an arbitrary motion the inertia tensor $I_{ij}^{(sp)}$ in space-fixed axes is not constant (unless, trivially, it is proportional to δ_{ij}). Show that in general the motion of a nontrivial body whose angular momentum $\mathbf{L}^{(sp)}$ is conserved (in a space-fixed frame) involves an angular velocity $\underline{\omega}$ which changes in time. [Hint: consider $L_i^{(sp)} = I_{ij}^{(sp)}\omega_j$.]

(iii) The problem of finding ω for a body of fixed angular momentum (not parallel to a principal axis) is best solved by adopting a body frame where the inertia tensor is constant. Show from (5) that *in the body frame* $\mathbf{L}$ obeys

$$d\mathbf{L}/dt = -\underline{\omega} \wedge \mathbf{L} \tag{6}$$

(iv) Deduce that

$$I_{ij}\,\dot{\omega}_j = -\epsilon_{ijk}\,\omega_j\,I_{kl}\,\omega_l$$

and show that, by taking as the body-fixed frame the principal frame of I_{ij}, this can be simplified to

$$I_1\,\dot{\omega}_1 = -\omega_2\,\omega_3(I_3 - I_2)$$

$$I_2\,\dot{\omega}_2 = -\omega_3\,\omega_1(I_1 - I_3)$$

$$I_3\,\dot{\omega}_3 = -\omega_1\,\omega_2(I_2 - I_1)$$

where I_1, I_2, I_3 are the principal moments of inertia.

(v) For the inertia tensor of the square frame found in question (i), $I_1 = I_2$ (such a body is known as a "symmetric top"). Prove for this case that $\underline{\omega}$ precesses in the body frame according to

$$\omega_3 = \text{const.} \;\; ; \;\; \omega_1 = A\cos(\Omega t) \;\; ; \;\; \omega_2 = A\sin(\Omega t)$$

where $\Omega = \frac{(I_3 - I_1)}{I_1}\omega_3$, and so confirm that $\underline{\omega}$ is not a constant vector unless it happens to point along one of the principal axes.

C. Stress and strain tensors.

C 1. Consider a small area element $d\mathbf{S}$ embedded at a point $\mathbf{r}$ in an isotropic liquid. If the material is subject to pressure P, there is a normal force on each side of the surface element $Pd\mathbf{S}$. These point in opposite directions, so the net force on the surface element is zero – which is as it must be, since the element has no mass and would otherwise be subject to infinite acceleration.

In the case of a solid, there can also be a *shear force* exerted in equal and opposite directions on the two sides of the element; such a force lies in the plane of the element rather than normal to it. In the general case, we can write

$$d\mathbf{F} = \underline{\underline{\tau}} \cdot d\mathbf{S} \tag{7}$$

where $\underline{\underline{\tau}}$ is the **stress tensor** at the point $\mathbf{r}$. For example the element τ_{xz} of $\underline{\underline{\tau}}$ determines the x-force per unit area on an element of surface with its normal pointing in the z-direction: $dF_x = \tau_{xz} dS_z$. For a surface element of general orientation $dF_x = \tau_{xx} dS_x + \tau_{xy} dS_y + \tau_{xz} dS_z$.

(i) Show that a small box of sides dx, dy, dz experiences a couple $(\tau_{xz} - \tau_{zx})dx\,dy\,dz$ about the y axis. The corresponding moment of inertia is of order $(dx)^2(dz)^2 dy$. Explain why this means that $\underline{\underline{\tau}}$ must be symmetric. Show also that, if $\underline{\underline{\tau}}(\underline{r})$ is a function of position, then the net force on a finite piece of material is given by

$$F_i = \int \frac{\partial \tau_{ij}}{\partial r_j}\, dV \quad . \tag{8}$$

where (in a conventional notation) $\frac{\partial}{\partial r_j} \equiv \nabla_j$.

(ii) **Strain:** Suppose a solid material is subjected to a small deformation $\mathbf{r} \to \mathbf{r} + \mathbf{u}(\mathbf{r})$. So long as the deformation depends continuously on position, we may write

$$du_i = \frac{\partial u_i}{\partial r_j}\, dr_j \quad . \tag{9}$$

Let us now decompose the tensor $U_{ij} \equiv \frac{\partial u_i}{\partial r_j}$ into its symmetric and anti-symmetric parts:

$$U_{ij} = e_{ij} + R_{ij}$$

$$e_{ij} = \frac{1}{2}\left(U_{ij} + U_{ji}\right)$$
$$R_{ij} = \frac{1}{2}\left(U_{ij} - U_{ji}\right)$$

Show that $\underline{\underline{e}}$ is symmetric and $\underline{\underline{R}}$ antisymmetric. Show also that $\underline{\underline{R}}$ corresponds to a pure rotation (i.e. without distortion) of the bit of matter near $\mathbf{r}$. [Hint: compare Eq.(4) in section A.] What type of deformation does $\underline{\underline{e}}$ represent? Why may we assume that the stress tensor $\underline{\underline{\tau}}$ depends on the **strain tensor** $\underline{\underline{e}}$, but has no dependence on $\underline{\underline{R}}$?

(iii) **Elasticity:** For small deformations, we expect the stress and strain to be linearly related, but since both are described by second-rank tensors, the relation between them generally involves a tensor of the fourth rank:

$$\tau_{ij} = E_{ijkl}\, e_{kl} \tag{10}$$

which is the generalization of Hooke's law to an arbitrary three-dimensional material. Since $\underline{\underline{\tau}}$ is symmetric, E_{ijkl} must be symmetric under exchange of i and j; various other symmetries can be found from considering the form of the stored elastic energy, but even when these are taken into account, E_{ijkl} has 21 independent components.

(iv) **Isotropic elastic solids** are much simpler however, and have only two independent constants. These are the Young's modulus Y and Poisson's ratio σ. The definition of these is that in the principal axes of τ_{ij}, the following relation applies

$$Y e_{xx} = \tau_{xx} - \sigma(\tau_{yy} + \tau_{zz}) \tag{11}$$

with similar relations for e_{yy}, e_{zz}; all off-diagonal elements of $\underline{\underline{e}}$ are zero in this coordinate system. By rearranging Eq.(11) obtain the result

$$Y e_{xx} = (1+\sigma)\tau_{xx} - \sigma\tau_{ii}$$

(where x indicies are not summed over but τ_{ii} is summed) with similar relations for yy and zz components. [Note that $\tau_{ii} = \mathrm{Trace}(\underline{\underline{\tau}})$ under the summation convention.] Show that, by adding Eq.(11) to its analogues,

$$\tau_{ii} = \frac{Y e_{ii}}{1-2\sigma}$$

Combine the above two results to give expressions for τ_{xx} etc in terms of the elements of $\underline{\underline{e}}$, and hence derive the following result in the principal frame:

$$\tau_{ij} = \frac{Y}{1+\sigma}\left(e_{ij} + \frac{\sigma}{1-2\sigma}\delta_{ij}\mathrm{Tr}(\underline{\underline{e}})\right).$$

[Bear in mind that $e_{ij} = 0$ for $i \neq j$, in this frame.] Hence, derive the general (frame independent) form for the elasticity tensor in an isotropic solid:

$$E_{ijkl} = \frac{Y}{1+\sigma}\left[\frac{\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}}{2} + \left[\frac{\sigma}{(1-2\sigma)}\right]\delta_{ij}\,\delta_{kl}\right]$$

C 2. (i) Apply E_{ijkl} to suitable deformations to recover expressions for the bulk modulus and shear modulus of an isotropic solid in terms of Y and σ.

(ii) Show that the displacement $\underline{u}$ inside a homogeneous isotropic solid, subject to external forces only at its surface, obeys

$$\mathrm{grad}\ \mathrm{div}\underline{u} + (1-2\sigma)\nabla^2\underline{u} = 0.$$

D. Transformation Laws.

D 1. Here we consider transformations of vectors and tensors between two sets of base vectors, each of which is orthonormal. We call our two sets of base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$, so that we can use suffix notation, e.g., the fact that the $\mathbf{e}$ basis is orthonormal can be written

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{12}$$

The unit vectors $\mathbf{e}'_i$ are linear combinations of the $\mathbf{e}_i$:

$$\mathbf{e}'_i = T_{ij}\mathbf{e}_j \ . \tag{13}$$

Suppose that $\mathbf{x} = x_i\mathbf{e}_i = x'_i\mathbf{e}'_i$ is any vector, and $\mathbf{y} = \underline{\underline{A}}\mathbf{x}$. Prove that

(i) $\mathbf{e}'_i \cdot \mathbf{e}_k = T_{ik}$

(ii) The inverse transformation $\mathbf{e}_i = T^{-1}_{ik}\mathbf{e}'_k$ has $T^{-1}_{ik} = T_{ki}$

(iii) $x'_i = T_{ik}x_k$

(iv) $x_i = T_{ki}x'_k$

(v) $y'_i = T_{ij}T_{lk}A_{jk}x'_l$

This means you have found the components of the tensor $\underline{\underline{A}}$ in the rotated coordinate system:

$$A'_{il} = T_{ij}T_{lk}A_{jk}$$

Indeed, this property under *orthogonal coordinate transformations* T is, mathematically, often taken as the defining property of a tensor $\underline{\underline{A}}$. For tensors of higher rank than 2, one factor of the transformation matrix T is needed for each index. Transformations with $\det(T) = 1$ are usually called *proper rotations* and those with $\det(T) = -1$, improper rotations. Show that these are the only possible values of the determinant for transformations between orthonormal bases as defined above.

Appendix: The antisymmetric three-tensor ϵ_{ijk}.

[Reprise of Sheet V section D.]

A tensor with two indices such as δ_{ij} is called a tensor of *second rank*, and its linear operation on a vector gives another vector. A vector is technically a tensor of rank one; a scalar of rank zero. Tensors of higher rank can also be defined. For example a tensor of rank four, Q_{ijkl} acts on a rank two tensor to give another rank two tensor, $B_{ij} = Q_{ijkl}A_{kl}$ (with summation over repeated indices). For tensors higher than rank two, dyadic notation becomes extremely cumbersome and suffix notation is essential.

1. An important and useful tensor of rank three is the *fully antisymmetric three tensor* denoted ϵ_{ijk}. This operates on either a second rank tensor or *two* vectors to give a vector. For $i \neq j \neq k$ it is defined as $\epsilon_{ijk} = 1$ if ijk is a cyclic permutation of x, y, z (e.g., yzx) and $= -1$ if the order is anticyclic (e.g., xzy). If any two of ijk are the same then $\epsilon_{ijk} = 0$ (e.g., $\epsilon_{xxy} = 0$). The tensor ϵ_{ijk} is very useful for writing down a vector product in suffix notation. Show that

$$(\mathbf{A} \wedge \mathbf{B})_i = \epsilon_{ijk}\, A_j\, B_k \tag{14}$$

The following relation

$$\epsilon_{ijk}\, \epsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl} \tag{15}$$

is tedious to prove (using components – you should come across this in maths lectures) but is worth remembering. It saves a great deal of time when manipulating vector identities such as

$$\underline{A} \wedge (\underline{B} \wedge \underline{C}) = \epsilon_{ijk}\, A_j\, \epsilon_{klm}\, B_l\, C_m$$

$$= A_j\, B_l\, C_m\, (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})$$

$$= B_i\, A_j\, C_j - C_i\, A_j\, B_j = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$

which is a familiar result normally derived by more tedious means.

The result Eq.(15) is especially helpful when dealing with vector identities involving the ∇ operator. Because ϵ_{ijk} is a constant tensor, it commutes with ∇, and expressions can be re-ordered accordingly. For example

$$\nabla \cdot (\underline{E} \wedge \underline{H}) \equiv \nabla_i (\epsilon_{ijk} E_j H_k) \tag{16}$$

$$= \epsilon_{ijk} \nabla_i (E_j H_k) = \epsilon_{ijk} [E_j \nabla_i H_k + H_k \nabla_i E_j] \tag{17}$$

$$= -E_j \, \epsilon_{jik} \nabla_i H_k + H_k \, \epsilon_{kij} \nabla_i E_j \tag{18}$$

$$= -\underline{E} \cdot \nabla \wedge \underline{H} + \underline{H} \cdot \nabla \wedge \underline{E} \tag{19}$$

which is a result used in in connection with Poynting's vector in electromagnetism.

(i) Make sure you understand *each step* of this derivation.

(ii) Show, using similar methods, that

$$\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \wedge (\psi \mathbf{A}) = \psi \nabla \wedge \mathbf{A} - \mathbf{A} \wedge \nabla \psi$$

These relations are useful in electromagnetism and other topics in the IB and parts II and III courses.