EXAMPLES CLASS IN MATHEMATICAL PHYSICS

IV. Spherical Harmonics in Physics (27.1.99)

Commentary: You have met the operator ∇^2 in a number of physical situations so far, for example diffusion, waves and electro- and magnetostatics. Cartesian coordinates were appropriate in these problems. In this sheet we examine the structure of ∇^2 in spherical polars, which are often the more natural coordinates, for instance when considering a sphere in a field or in scattering problems. The mathematical material is introduced as needed, but some of the derivations are left to the Appendix. We shall be particularly interested in the *angular behaviour* of solutions of Laplace's equation $\nabla^2\phi = 0$ (known as the Legendre functions). Each of these angular solution is associated with a pair of specific radial solutions, whose physical interpretation is perhaps most obvious in the context of electrostatics problems.

In Section A, therefore, we consider two electrostatics problems which can be solved using Legendre polynomials. One of these you will have met before (a sphere in an applied field); the purpose here is to show that a properly systematic approach exists for problems of this kind. Section B shows that for cylindrical problems the angular part of Laplace's equation involves sines and cosines; this makes a connection between angular problems and the more familiar concepts of Fourier theory. Section C applies the idea of multipoles to a problem in fluid mechanics, which is probably unfamiliar but which has a deep mathematical analogy with electrostatics.

A. Fundamentals

Laplace's equation, $\nabla^2 V = 0$, reads in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} V + \frac{1}{r^2}\Lambda V = 0 \tag{1}$$

where Legendre's operator Λ contains the angular variation with respect to θ and ϕ. For an axially symmetric problem, there is no ϕ dependence and Λ obeys

$$\Lambda V = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) \tag{2}$$

[See the Appendix for the general ϕ-dependent case.]

It turns out (see Appendix) that there is a complete set of separable solutions for V in the form $V = R_l(r) S_l(\theta, \phi)$ where the radial equation is

$$\frac{d}{dr} r^2 \frac{d}{dr} R = l(l+1) R \tag{3}$$

and the angular equation is

$$\Lambda S_l(\theta, \phi) = -l(l+1) S_l(\theta, \phi)\ . \tag{4}$$

In these equations, l is an integer.

The general solution of the radial equation is $R(r) = A_l\, r^l$ and $R(r) = B_l\, r^{-(l+1)}$ where A_l and B_l are constants. The solution, $S_l(\theta, \phi)$, of the angular equation is called a *Spherical Harmonic* of order l. When there is no ϕ-dependence, as in all the problems studied here, this can be written as $P_l(\cos\theta)$ where P_l is the l^{th} *Legendre Polynomial.* The complete general solution of Laplace's equation with axial symmetry is then

$$V(r, \theta) = \sum_{l=0}^{\infty} [A_l\, r^l + B_l\, r^{-(l+1)}] P_l(\cos\theta) \tag{5}$$

In the Appendix, you are invited to confirm some steps in these derivations, and investigate certain other mathematical properties of the Legendre polynomials; for example the P_l are a *complete* set of orthogonal functions. However, it is more important to be able to *apply* Eq.(5) to physical problems. To do this, you will need to know the first few Legendre Polynomials, which are:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2 - 1)/2, \quad P_3(x) = (5x^3 - 3x)/2.$$

Note that Legendre polynomials are normalised, by convention, to $P_l(1) = 1$.

A1. Electrostatics. Consider an electrostatic potential which obeys Laplace's equation outside some small region where charges reside. Show that for suitable choices of constants we can write (5) as:

$$V(r,\theta) = V_0 - E_1 r\cos\theta + E_2 r^2 P_2(\cos\theta) + \ldots + \frac{Q}{4\pi\epsilon_0 r} + \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + \frac{qP_2(\cos\theta)}{4\pi\epsilon_0 r^3} + \ldots \quad (6)$$

The interpretation of these terms is interesting, for example the E_1 term corresponds to a uniform field in the z direction (since $r\cos\theta = z$ is then the distance travelled in the direction of the field), and Q is an effective point charge. What are p and q? In general the coefficients of the negative powers of r are *multipole moments* of the localized charge distribution.

A2. Polarization of a sphere. "Why does an uncharged metal sphere of radius a in a field E_o uniform at ∞ appear to act like a dipole? What is its polarisability?"
To answer this,
(i) Write down a general solution for the potential outside the sphere, ie $V(r,\theta)$ for $r \geq a$. (Why are some terms of (6) unacceptable?)
(ii) The potential $V(a,\theta)$ is constant at the surface of the sphere. Use this fact to work out the final form for the potential outside the sphere.
[Ans: $V(r,\theta) = -E_o r\cos\theta + \frac{E_o a^3\cos\theta}{r^2}$ + const.]
(iii) Interpret the second term. Deduce the polarisability of the sphere.
(iv) Had the metal sphere instead been a dielectric sphere, what kind of field would we have found in the interior region, $r < a$?

Note the method that has been used. We have fixed the *angular* part of $V(r,\theta)$ by expanding $V(r=a,\theta)$ in the functions $P_l(\cos\theta)$. This determines the set of coefficients A_l and/or B_l. Which of these are required, depends on the boundary conditions. In this example, all the boundary conditions could be satisfied using only a few terms, in which case the uniqueness theorem states that all other coefficients must vanish. Such simple problems can often be solved just by guessing which terms are present (e.g., the trick of placing an imaginary dipole at the sphere's centre). However, there are many other problems (such as the next one) where the full machinery is needed.

A3. Field inside a spherical capacitor. "A thin hollow metal sphere of radius $r=a$ is cut into hemispheres insulated from each other along the equator. The upper hemisphere is at potential V_0, the lower at $-V_0$. Find a general expression for the potential $r,x)$ within the shell, and calculate the field at the centre."

To solve this problem we must decompose the (unusual) angular dependence imposed at the boundary into its spherical harmonic components. This proceeds in close analogy with the Fourier methods described in previous sheets. To carry out the angular analysis, we will need two more facts about Legendre polynomials:

(a) The P_l's satisfy an orthogonality relation:

$$\int_{-1}^{+1} dx\, P_l(x)\, P_n(x) = \delta_{ln}\,\frac{2}{(2l+1)}. \tag{7}$$

(b) Spherical harmonics of odd order change sign under inversion of coordinates, that is, in going from a point on the unit sphere to the antipodes. This implies (for the case of no ϕ-dependence) that

$$P_l(-x) = (-1)^l\, P_l(x) \tag{8}$$

To solve the problem, proceed as follows:

(i) At fixed $r=a$, $V(r=a,\theta)$ is a function of the single variable θ. Plot the potential

$V(a,\theta) \equiv f(\cos\theta)$ against the variable $x \equiv \cos\theta$, where $-1 \le x \le +1$. Since the P_l are complete, this can be expanded as

$$f(x) = V_0 \sum_l b_l \, P_l(x) \; . \tag{9}$$

Use the orthogonality of the Legendre polynomials to find a general formula for the coefficients b_l. Confirm that $b_1 = 3/2$ and $b_3 = -7/8$.

(ii) Hence show that the potential inside the sphere obeys

$$V(r,\theta) = \sum_{l=1,3,5,\ldots} A_l r^l P_l(\cos\theta) \tag{10}$$

where $A_l = V_0 a^{-l}(2l+1) \int_0^1 P_l(x) dx$. [Hint: We know $V(a,\theta)$ and hence $A_l a^l = b_l$ gives us A_l.]

(iii) Calculate the electric field at the centre of the sphere. [Think carefully about which terms in (10) can contribute, and any symmetry arguments that can be used to simplify the problem. Ans: $3V_0/2a$, axial.]

B. Laplace's Equation in Two Dimensions

Laplaces's equation in 2-D, appropriate to systems of cylindrical symmetry, is:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Phi + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\Phi = 0.$$

The angular part is simple, and the 2-D equivalents of the Legendre functions are:

$$\sin n\phi \; , \quad \cos n\phi$$

with corresponding radial functions $\frac{1}{r^n}, r^n$ (and $\ln(r)$ for $n = 0$). These combinations give the 2-D multipoles equivalent to those enumerated in A(1). The potential of a uniform field along the x direction ($\phi = 0$) is of course $V = -Er\cos\phi$.

B 1. What is the (r, ϕ) dependence of the potential and field of a line charge? Use Gauss's theorem to confirm your result. A *line dipole* is the limit of two equal and opposite line charges close together. Show that the corresponding potential is $V = \frac{p \cos\phi}{2\pi\epsilon_0 r}$ where p is the dipole moment per unit length.

An uncharged metal cylinder of radius a in a field E_o perpendicular to its axis and uniform at ∞ appears to act as a line dipole. What is its dipole moment per unit length? To answer this repeat A2(i)-(iv) with the word cylinder replacing sphere. The paragraph at the end of A(2) about fixing the angular functions still holds and question A(3) could also be posed for a cylinder slit into 2 half cylinders held at different potentials.

C. Fluid Mechanics.

You are not expected to be familiar with the physics of this section. However, you should be able to make progress using the same techniques as before.

C1. Irrotational flow in a fluid is where the velocity field $\mathbf{v}(\mathbf{r})$ is "curl-free", that is $\nabla \wedge \mathbf{v} = 0$. We can then write $\mathbf{v} = -\nabla V$; where $V(\mathbf{r})$ is a potential function. Show that for an *incompressible* fluid, the velocity potential V satisfies $\nabla^2 V = 0$. This gives an analogy with electrostatics, useful for problems in fluid motion.

(i) What is the potential associated with a uniform flow $\mathbf{u}$ parallel to the z axis?

(ii) A sphere of radius a is held stationary in such a flow field. What is the flow field now? [Hint: write down the general form of the potential and select terms that will allow you to satisfy suitable boundary conditions at $r = a$.] Sketch the flow field and interpret this physically.

(iii) Imagine instead a sphere moving with velocity $\mathbf{u}$ through a stationary fluid of density ρ. How does the above solution change? Show that the kinetic energy of the fluid set in motion by the sphere is $\frac{1}{3}\pi\rho a^3 u^2$. Interpret this result, and use it to find the acceleration with which a ping pong ball of negligible mass rises when released under water.

Appendix. Background Theory and Mathematical Exercises

(i) Confirm that the radial equation (3) has solutions $R(r) = A_l\, r^l$ and $R(r) = B_l\, r^{-(l+1)}$ where A_l and B_l are constants.

(ii) The general solution $S_l(\theta, \phi)$ of (4) is itself separable as $\Theta(\theta)\, \Phi(\phi)$. Let us consider first the situation where there is no ϕ dependence, that is $\Phi = \text{const}$. Show that under the change of variable $x = \cos(\theta)$, $y(x) = \Theta(\theta)$, Equation (4) then reduces to

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \tag{11}$$

which is **Legendre's Equation**. The separation constant is by convention written $l(l+1)$, since (as it turns out) only for l an integer is there a solution $y(x)$ finite over the range $(-1, +1)$ of $x \equiv \cos\theta$. The finite solutions are polynomials, the *Legendre Polynomials* $y(x) = P_l(x)$.

(iii) Where there is a dependence on the azimuthal angle ϕ, the full form of the Legendre operator is:

$$\Lambda V = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}. \tag{12}$$

The Θ and Φ dependences can also be separated into two equations, with separation constant m^2. This yields the following equation for Φ:

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{13}$$

with solution $e^{im\phi}$. Why is m an integer?

The Θ equation is now slightly more complicated and yields the *Associated Legendre Functions* $P_l^m(x)$ satisfying:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + [l(l+1) - \frac{m^2}{(1 - x^2)}]y = 0 \tag{14}$$

where the previous constant remains of the form $l(l+1)$, with l an integer (in order to have finite solutions). There are $2l+1$ independent solutions of this equation, which may be written $P_l^m(x)e^{im\phi}$ with $m = -l, ..0.., +l$. This set is normally used in Quantum Mechanics. In other applications, real ϕ solutions are often used instead, that is:

$P_l^m(x)\cos m\phi$, with $m = 0, 1, \ldots, l$

$P_l^m(x)\sin m\phi$, with $m = 1, \ldots, l$

A general solution to Laplace's equation with ϕ-dependence is:

$$\sum_{l=1}^{\infty} A_{l,m} P_l^m(\cos\theta)e^{im\phi}\ r^l + \sum_{l=0}^{\infty} \frac{B_{l,m}P_l^m(\cos\theta)e^{im\phi}}{r^{l+1}} \tag{15}$$

The Associated Legendre functions can be derived from the *Rodrigues* relation:

$$P_l^m(x) = \frac{(1-x^2)^{m/2}}{2^l(l+m)!}(\frac{d}{dx})^{l+m}(x^2-1)^l \tag{16}$$

These functions are especially important in atomic physics where $l(l+1)$ and m are the quantum numbers for the operators $L^2 = \mathbf{L}\cdot\mathbf{L}$ and L_z, where $\mathbf{L}$ is the angular momentum operator. The completeness and orthogonality of the spherical harmonics follows from the fact that these operators are hermitian.

(iv) Show that if $\psi_l = r^l S_l(\theta,\phi)$ and $\psi_n = r^n S_n(\theta,\phi)$ both obey Laplace's equation, then S_l and S_n are *orthogonal* in the following sense:

$$\int S_l S_n\, d\Omega = 0 \tag{17}$$

where $d\Omega$ is the solid angle element. [Hint: use Green's theorem $\int_V (f\nabla^2 g - g\nabla^2 f)d^3r = \int_S (f\nabla g - g\nabla f)\cdot d\mathbf{S}$, integrating over a spherical region V.]

(v) Show that for Legendre polynomials (no ϕ variation) this implies

$$\int_{-1}^{+1} P_l(x)\, P_n(x)\, dx = 0 \qquad (l \neq n) \tag{18}$$

Hence Legendre polynomials are *orthogonal*; they are also *complete*, so that any function $f(\theta)$ with θ defined in the range $(0,\pi)$ can be expanded in terms of them. This is the basis of the approaches to potential problems used in Sections A and B (and equivalently in C with sin and cos which are also complete).