EXAMPLES CLASS IN MATHEMATICAL PHYSICS

VI. The Structure of Electromagnetic Theory (10.2.99)

Commentary: Maxwell's equations are usually represented as the synthesis of more elementary laws – those of Coulomb, Ampere, Lenz and Faraday. However, it is possible to take the opposite approach: to start by postulating Maxwell's equations as relations among various vector and scalar fields ($\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}, \rho$) and then use mathematical analysis (with the minimum number of further postulates) to identify the physical significance of these quantities. We give a flavour of this approach here by starting from Maxwell's equations and deriving many of the other formal results of electromagnetism. The sheet therefore mainly consists of bookwork but this is an excellent way of strengthening important 1B lecture material.

Section A covers electro- and magnetostatics of charges and currents *in vacuo*; Section B is about the response of dielectric and magnetic materials; and Section C concerns the energy stored in electric and magnetic fields. Section D offers food for thought to those who like to ponder on "how things might have been".

The formal approach to electromagnetism is not simply a mathematical exercise (though much intuition for other, unrelated problems can be gained by it). From the equations emerge new physical possibilities *unanticipated* from the elementary laws, for instance Maxwell's prediction of electromagnetic waves and the emission of energy by accelerating charges. Similar developments arise in quantum mechanics, relativity and particle physics. Thus mathematics gives a formal coherence to physical theory and allows predictions to be made beyond the realm of existing experiment.

A. Time-independent fields

A1. We assume Maxwell's equations in the usual form

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{3}$$

$$\nabla \wedge \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{4}$$

Show that the continuity equation (for electric charge) follows from Maxwell's equations:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \tag{5}$$

and interpret this physically.

A2. Show that if the fields are *time-independent* then $\mathbf{E}$ is conservative, that is for any closed circuit

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{6}$$

Hence in this case (as we saw in Sheet I) $\mathbf{E}$ can be written in the form $\mathbf{E} = -\nabla V$, V being the (scalar) electrostatic potential. Show that $\mathbf{E}$ cannot be so expressed if $\mathbf{B}$ is time-varying. Show also that, apart from a constant, $V(\mathbf{r})$ is the work done in bringing unit charge from infinity to the point $\mathbf{r}$ (the usual formula for the force acting on an electric charge may be assumed).

A3. Show that in any static problem, the field $\mathbf{H}$ can be expressed in the form $-\nabla V_{mag}$, when permanent magnets and magnetisable material are present but there are *no* currents. If there *are* steady currents show that

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enclosed}. \tag{7}$$

This equation is often used for time-varying fields. Under what conditions is this a good approximation? If in doubt, look at the supplementary question at the end of this sheet.

A4. Maxwell's equations need to be supplemented by *constitutive relations* connecting **D** and **E**, and **B** and **H**. In vacuo these are

$$\mathbf{D} = \epsilon_0 \mathbf{E} \tag{8}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{9}$$

Note that these relations *cannot* be predicted from Maxwell's equations alone. Derive from Maxwell's equations and the constitutive relations the Poisson equation for an electrostatic potential when charges are present in the vacuum:

$$\nabla^2 V = -\rho/\epsilon_0 \tag{10}$$

(i) Hence derive the electrostatic potential for a point charge Q at the origin (Coulomb's law)

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r} \tag{11}$$

(ii) Show also that the potential at $\mathbf{r}$ due to an arbitrary charge density $\rho(\mathbf{r}')$ is

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{12}$$

(iii) Derive the potential for an electric point dipole $\mathbf{p}$ at the origin

$$V(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \mathbf{p}.\nabla\frac{1}{r} \tag{13}$$

[Hint: use the Taylor series for a function of a vector, $f(\mathbf{a} + \mathbf{b}) = \mathbf{f}(\mathbf{a}) + \mathbf{b} \cdot \nabla \mathbf{f}(\mathbf{a}) + O(\mathbf{b}^2)$.]

B. Polarisation and Magnetisation

When media are present, the constitutive relations become

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{14}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{15}$$

where $\mathbf{P}$ and $\mathbf{M}$ are interpreted physically as the polarisation (that is, the electric dipole density) and the magnetisation (magnetic dipole density). One can view (8) and (9) as the fundamental equations, and then *derive* (14) and (15) from them using Maxwell's equations, by calculating the effects of polarisation or magnetisation induced in the material. Below we follow this through for the electric case.

B1. A region τ has a distribution of polarisation $\mathbf{P}(\mathbf{r})$ within it. Write down an integral expression for the electrostatic potential it produces. Show that the potential is identical to that produced by a charge distribution $\rho(\mathbf{r}') = -\nabla \cdot \mathbf{P}$ at points $\mathbf{r}'$ in the interior of τ, together with a distribution of surface charge $\mathbf{P} \cdot \mathbf{n}$ over the surface of τ. [Hint: you will need the results of Q. A4. Also recall that $\nabla \cdot (\mathbf{A}f) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$.]

B2. We have just seen that we can think of the potential V due to a polarization density as having instead been produced by an equivalent distribution of fictitious charges. Now consider a distribution of free charge ρ and polarisation present together. Replace the polarisation by its equivalent fictitious charges and derive an expression for $\nabla^2 V$. Show that Maxwell's equation (3) $\nabla \cdot \mathbf{D} = \rho$ holds, *provided* that $\mathbf{D}$ is defined as in (14). [**Remark**: this shows the utility of $\mathbf{D}$ over $\mathbf{E}$ when media are present – it is $\mathbf{D}$ that is related to the *free* charges ρ in the problem, for instance the actual charge stored on the plates of a capacitor. $\mathbf{H}$ plays the same useful role in magnetism in that it relates to the *free* currents.]

Under what conditions can Eqs.(14) and (15) be replaced by the following?

$$\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E} \tag{16}$$

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} \tag{17}$$

B3. Boundary Matching Problems: Across any sharp boundary between two media the values of the fields can change discontinuously.

(i) By taking suitable integrals of Maxwell's equations in the neighbourhood of the surface show that (in the absence of free surface charges and currents) the following are continuous:

(a) the normal components of $\mathbf{B}$ and $\mathbf{D}$;

(b) the tangential components of $\mathbf{E}$ and $\mathbf{H}$.

Show that all remaining components of the fields are *discontinuous* at a boundary where ϵ_r and μ_r both change suddenly.

(ii) We see from the above that the "boundary conditions" at the junction between two media can be derived from Maxwell's equations and are not extra assumptions. By using the relations (16) and (17), one could instead eliminate at the outset two of the fields $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$ from Eqs.(1-4) and solve the differential equations for the remaining pair (say, $\mathbf{E}$ and $\mathbf{B}$). What mathematical difficulties would this pose when considering the boundary between two polarizable media? [Hint: consider, for example, the behaviour of $\mathrm{div}(\epsilon_r \epsilon_0 \mathbf{E}) = \rho$ at such a boundary.]

(iii) If free surface charges σ and surface currents $\mathbf{j}_s$ are present, derive the modified equations for boundary matching problems

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \tag{18}$$

$$\mathbf{n} \wedge (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j}_s \tag{19}$$

where $\mathbf{n}$ is the unit normal pointing from region 1 to region 2.

C. Fields and their Energy

C1. Starting from Maxwell's equations, derive Faraday's Law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt \tag{20}$$

where $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$ is defined to be the flux of the field $\mathbf{B}$ through a stationary circuit C. In this formulation we have used the *total* field whereas the usual statement of Faraday's Law refers only to the field in the circuit induced by $\dot{\mathbf{B}}$. Explain this apparent discrepancy.

C2. The electromotive force acting on a charge Q moving with velocity $\mathbf{v}$ is given by the Lorentz force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{21}$$

(i) What is the rate of transfer of energy W from the electromagnetic fields to the charge Q (that is, their rate of doing work on the particle)? How does this generalize to a continuous current distribution $\mathbf{j}$ in a region τ ? [Ans: $W = \int \mathbf{j} \cdot \mathbf{E}\, d\tau$.]

(ii) The energy stored in electromagnetic fields is:

$$U = \int d\tau \left[\int_0^{\mathbf{D}} \mathbf{E} \cdot d\mathbf{D} + \int_0^{\mathbf{B}} \mathbf{H} \cdot d\mathbf{B} \right] \tag{22}$$

What does this become if the relationship between the $\mathbf{D}$ and $\mathbf{E}$ fields is linear?

(iii) Show from Maxwell's equations that

$$-\mathrm{div}(\mathbf{E} \wedge \mathbf{H}) = \mathbf{j} \cdot \mathbf{E} + \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} \tag{23}$$

[Hint: you will need the relation $\nabla \cdot (\mathbf{g} \wedge \mathbf{f}) = \mathbf{f} \cdot (\nabla \wedge \mathbf{g}) - \mathbf{g} \cdot (\nabla \wedge \mathbf{f})$.]

(iv) Hence derive the relationship

$$dU/dt = -\int d\tau(\mathbf{j} \cdot \mathbf{E}) - \int (\mathbf{E} \wedge \mathbf{H}) \cdot d\mathbf{S} \tag{24}$$

for the rate of change of field energy when a current distribution $\mathbf{j}(\mathbf{r})$ is present. What is the meaning of the second term?

(v) In the case of linear relationships between $\mathbf{E}$ and $\mathbf{D}$ and between $\mathbf{B}$ and $\mathbf{H}$, show that the first term in (22) can be written as $\frac{1}{2} \int d\tau(\rho(\mathbf{r})V(\mathbf{r}))$ and the second as $\frac{1}{2} \int d\tau(\mathbf{A}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}))$, where $\mathbf{A}$ is the magnetic vector potential. Show further that in the case where all the magnetic field is generated by a collection of closed circuits carrying currents I_n and threaded by fluxes Φ_n, the second term reduces to $\frac{1}{2} \sum_n I_n \Phi_n$.

D. What if?

D1. According to some recent theories of fundamental interactions, magnetic monopoles should exist. Supposing such particles were discovered, Eq.(2) would become:

$$\nabla \cdot \mathbf{B} = \lambda \tag{25}$$

where λ is the magnetic monopole density (defined in some suitable units).

(i) Write down a continuity equation for the monopole "current density", $\mathbf{f}$ (cf. Eq.(5)).

(ii) Show that, Eq.(1) must be replaced by

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{f} \tag{26}$$

[Hint: consider flow of magnetic current onto a magnetic "capacitor"; apply Stokes' theorem to two different surfaces whose boundary is a loop encircling the current.]

(iii) Confirm that the remaining two Maxwell equations remain unaltered.

(iv) *(For discussion)* Which of the remaining equations on the earlier parts of this sheet would need to be modified, and how?

E. A Problem on Time-Varying Fields.

Consider a long solenoid of circular cross-section and radius R with n turns per unit length carrying current I. Inside the solenoid, (7) gives an axial field of strength $H = nI$. Show now that if I is slowly varying with time, an E field arises of strength $E(r) = \mu_o n \dot{I} r/2$ where r is the distance from the axis of the solenoid, and the field is in a tangential direction. Hence use (4) to show that the magnetic field at the centre of the solenoid is now given by $H = nI(1 - \epsilon_o \mu_o \ddot{I} R^2/4I)$. If I oscillates with angular frequency ω, show that the correction term remains negligible, as ω is increased, until the wavelength of light at frequency ω becomes comparable to the radius of the solenoid.