Question 20

Thomas - Fermi

It helps that you remember these fourier transform pairs (N.B - 3 Dimensions)

$$\frac{1}{r} \longleftrightarrow \frac{4\pi}{q^2} \qquad (1)$$

$$\frac{1}{r} e^{-q_0 r} \longleftrightarrow \frac{4\pi}{q^2 + q_0^2} \qquad (2)$$

Eqn 9.17 gives

$$\delta n(q) = \frac{v_{ext}(q)}{\frac{4\pi e^2}{q^2}\left[1 + \frac{q^2}{q_{TF}^2}\right]}$$

but (1) $\Rightarrow$ $v_{ext}(q) = \frac{-4\pi Q e}{q^2}$ (Charge $+Q$)

$$\therefore \delta n(q) = \frac{-Q/e}{(1 + q^2/q_{TF}^2)}$$

Hence (2) $\Rightarrow$

$$\delta n(r) = \frac{-Q/e}{(2\pi)^3} \int d^3q\, e^{i\underline{q}\cdot\underline{r}} \frac{q_{TF}^2}{q^2 + q_{TF}^2}$$

$$\propto -\frac{Qe}{r} e^{-q_{TF} r}$$

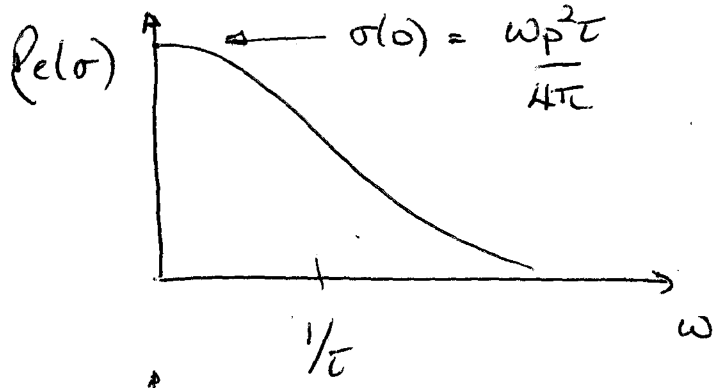
Note: (1) Perfect screening.

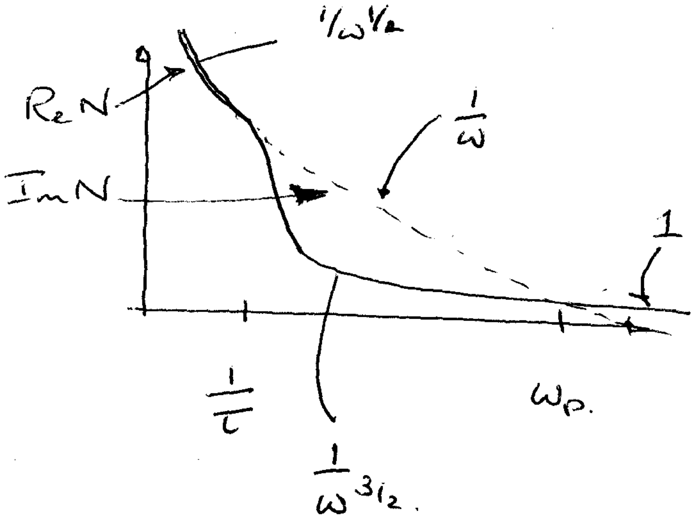
$$\int dr\, \delta n(r) = \int dr \int dq\, e^{i\underline{q}\cdot\underline{r}} \delta n(q) = \delta n(q=0) = -Q/e$$

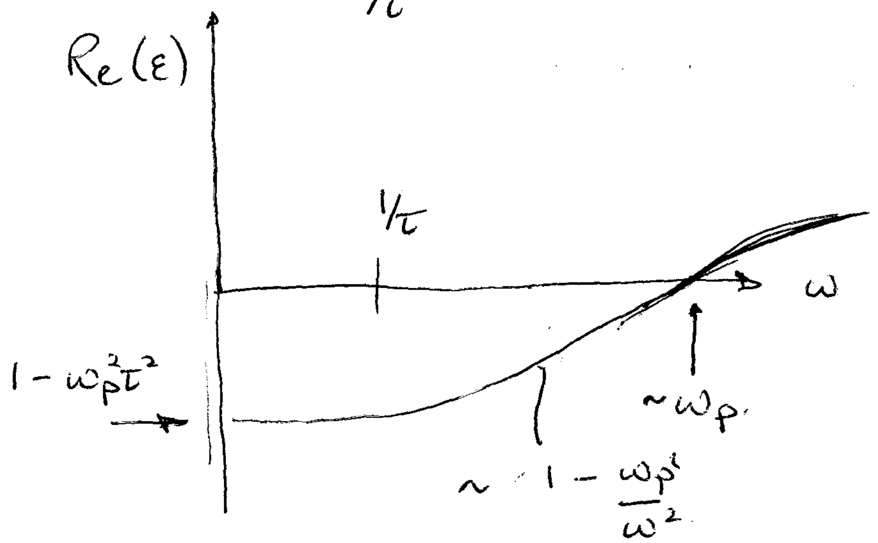
(2) Singular behaviour of δn as $r \to 0$, clearly wrong.

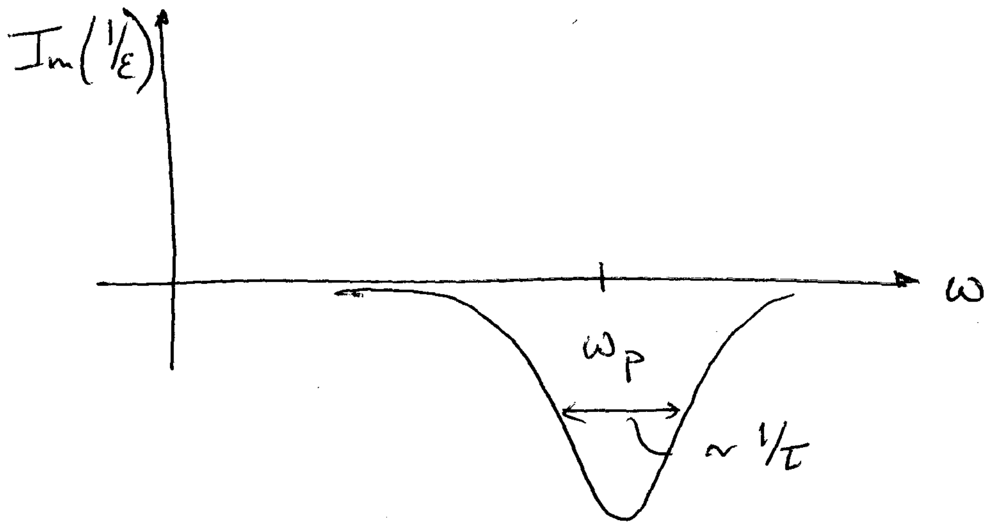
$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + (i/\tau)\omega} = 1 + \frac{4\pi\sigma i}{\omega}.$$

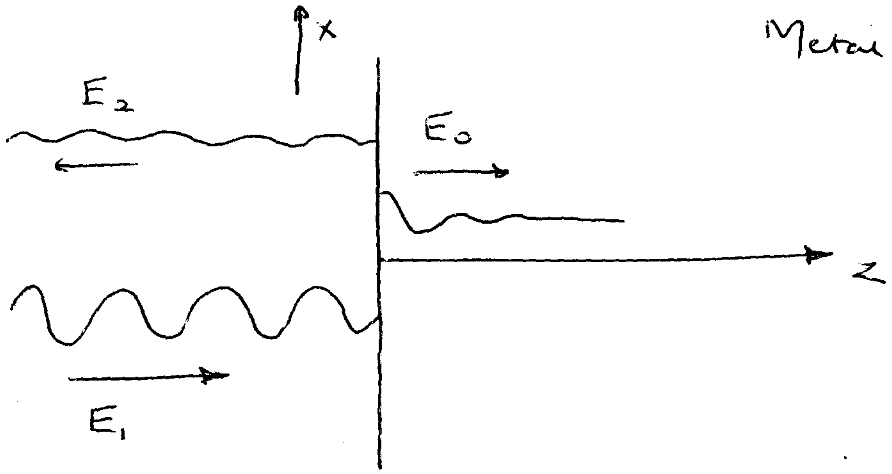
Assume a "good" metal. $\omega_p \tau \gg 1$.











In vacuum.

$$E_x = E_1 e^{i\omega\left(\frac{z}{c}-t\right)} + E_2 e^{-i\omega\left(\frac{z}{c}+t\right)}$$

In metal.

$$E_x = E_0 e^{i\omega\left(\frac{Nz}{c}-t\right)}.$$

Boundary conditions.

on E_x: $\Rightarrow$ $E_0 = E_1 + E_2$

Determine H_y from. $\nabla \times E = -\frac{\partial B}{\partial t}$ $\quad B = \mu_0 H$.

$\Rightarrow$ on H_y: $\Rightarrow$. $N E_0 = E_1 - E_2$

Thus. $R = \left|\frac{E_2}{E_1}\right|^2 = \left|\frac{1-N}{1+N}\right|^2$

$N = \sqrt{\varepsilon}$.

Various limits. $\left[\text{assume } \frac{1}{\tau} \ll \omega_p\right]$

(i) $\omega \ll 1/\tau$: $\quad \varepsilon \simeq i\frac{\omega_p^2 \tau}{\omega} = i\frac{4\pi\sigma(0)}{\omega}$

So. $N = (1+i)\left(\frac{2\pi\sigma(0)}{\omega}\right)^{1/2}$ $\quad \left[\text{In S.I. } (1+i)\left(\frac{\sigma}{2\omega\varepsilon_0}\right)^{1/2}\right]$.

Since $|N| \gg 1$

$$R = \left|\frac{1-1/N}{1+1/N}\right|^2 \simeq 1 - 2\left(\frac{1}{N}+\frac{1}{N^*}\right)$$

$$= \underbrace{1 - 2\left(\frac{\omega}{2\pi\sigma}\right)^{1/2}}_{\text{small deviation from unity}}$$

(ii) $\frac{1}{\tau} \ll \omega \ll \omega_p$.

Here $$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2(1 + i/\omega\tau)}$$

$$\simeq -\frac{\omega_p^2}{\omega^2}\left(1 - \frac{i}{\omega\tau}\right) + O(1)$$

$$\therefore N \cong i\left(\frac{\omega_p}{\omega}\right)\left(1 - \frac{1}{2}\frac{i}{\omega\tau}\right).$$

— Note that N is ~~p~~ nearly pure imaginary (which means $R \approx 1$) and $|N| \gg 1$

So as before

$$R \simeq 1 - 2\left(\frac{1}{N} + \frac{1}{N^*}\right)$$

$$\simeq 1 - 2.\frac{(N + N^*)}{|N|^2}$$

$$= 1 - 2.\frac{(\omega_p/\omega^2\tau)}{(\omega_p^2/\omega^2)} = 1 - \underbrace{\frac{2}{(\omega_p\tau)}}_{\text{still small, but constant in freq.}}$$

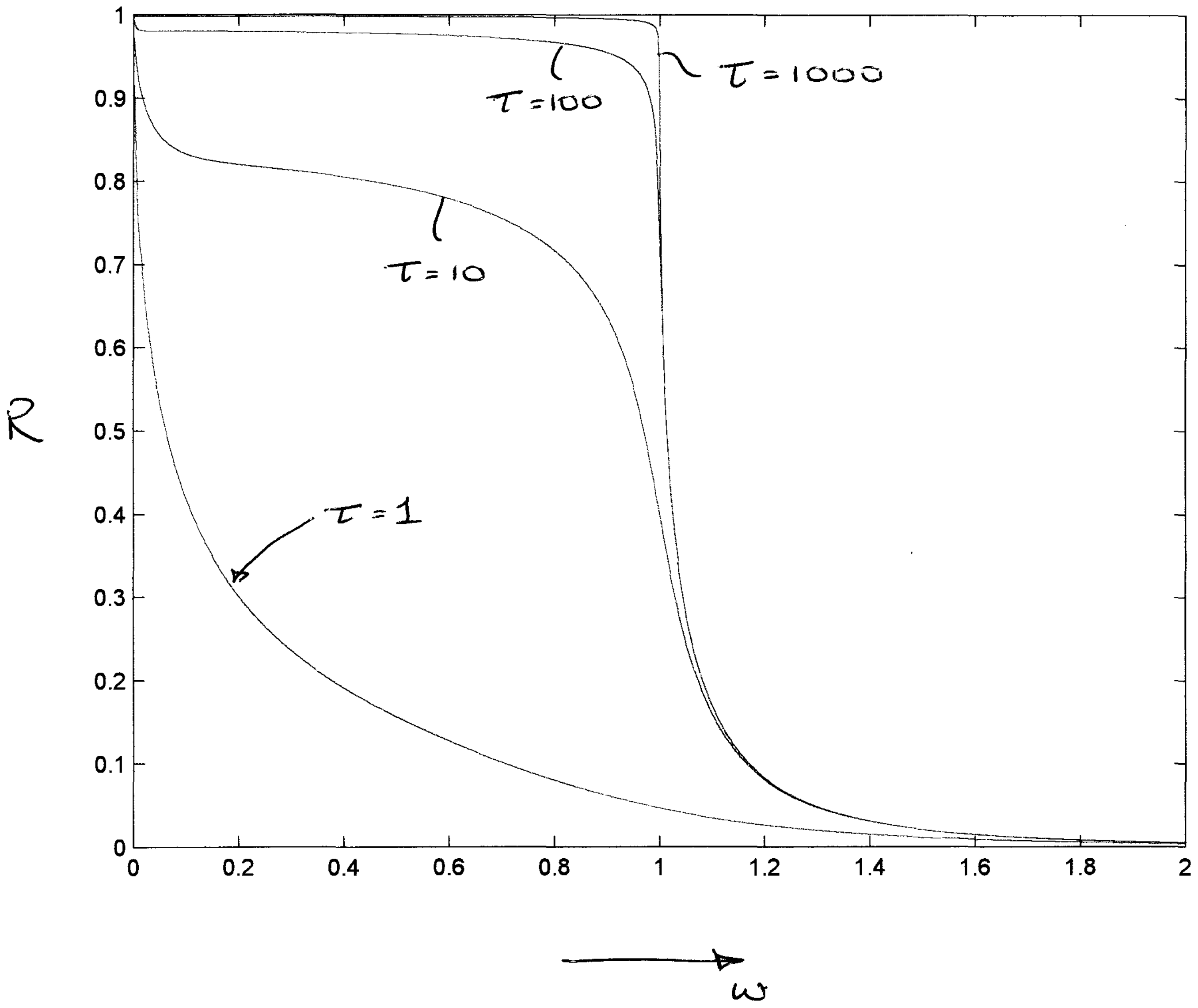
(iii) $\omega_p \ll \omega$.

Now ε positive, real, and close to 1

$$N \sim 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$$

$$R = \left|\frac{1 - N}{1 + N}\right|^2 \sim \left|\frac{\frac{1}{2}\frac{\omega_p^2}{\omega^2}}{2}\right|^2 \simeq \underline{\frac{1}{16}\left(\frac{\omega_p}{\omega}\right)^4}$$

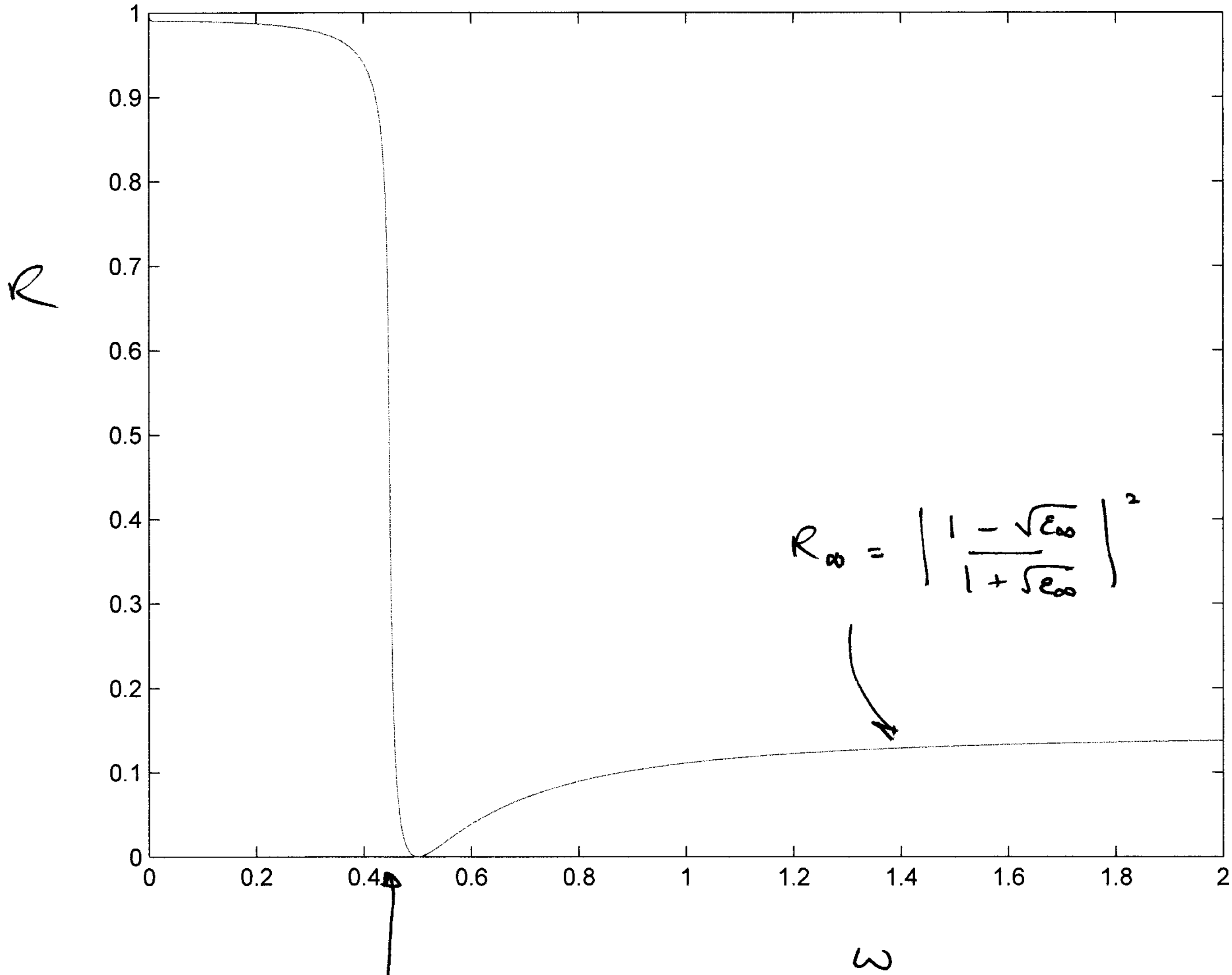
$\omega_p = 1$


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
R
τ = 1000
τ = 100
τ = 10
τ = 1
ω

It can be the case that the dielectric constant is not 1 at high frequencies — for example in a (metallic) doped semiconductor where

$$\varepsilon = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\omega/\tau}$$

Here is a plot for $\varepsilon_\infty = 5$, $\omega_p = 1$, $1/\tau = 5\times10^{-3}$



$$R_\infty = \left| \frac{1 - \sqrt{\varepsilon_\infty}}{1 + \sqrt{\varepsilon_\infty}} \right|^2$$

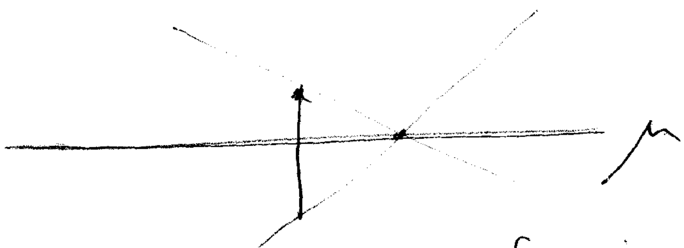
Plasma frequency is actually $\dfrac{\omega_p^2}{\varepsilon_\infty}$

For discussion in supervision, but briefly.

- Metals are shiny due to reflection
  - $\varepsilon < 0$; n purely imaginary
  - little absorption because conductivity is large

- Glass is transparent because there is a gap between valence & conduction bands
  - but glass is not a periodic crystal.

- Graphite black because although a metal, there is absorption by low energy bands



  - poor metal because fermi surface is "points" in k-space

- Sugar is white because of scattering of light — no absorption & little reflection at any particular surface.

Question 23

Include :

- Existence of fermi surface

- Screening in metal

$$\frac{1}{r} \longrightarrow \frac{e^{-q_{TF} r}}{r}$$

— in insulator $\frac{1}{r} \longrightarrow \frac{1}{\varepsilon r}$.

- Plasmons $\rightarrow$ $\varepsilon \sim 1 - \frac{\omega_p^2}{\omega^2}$

so $n = \sqrt{\varepsilon}$ pure imaginary below plasma freq. $\rightarrow$ good reflector

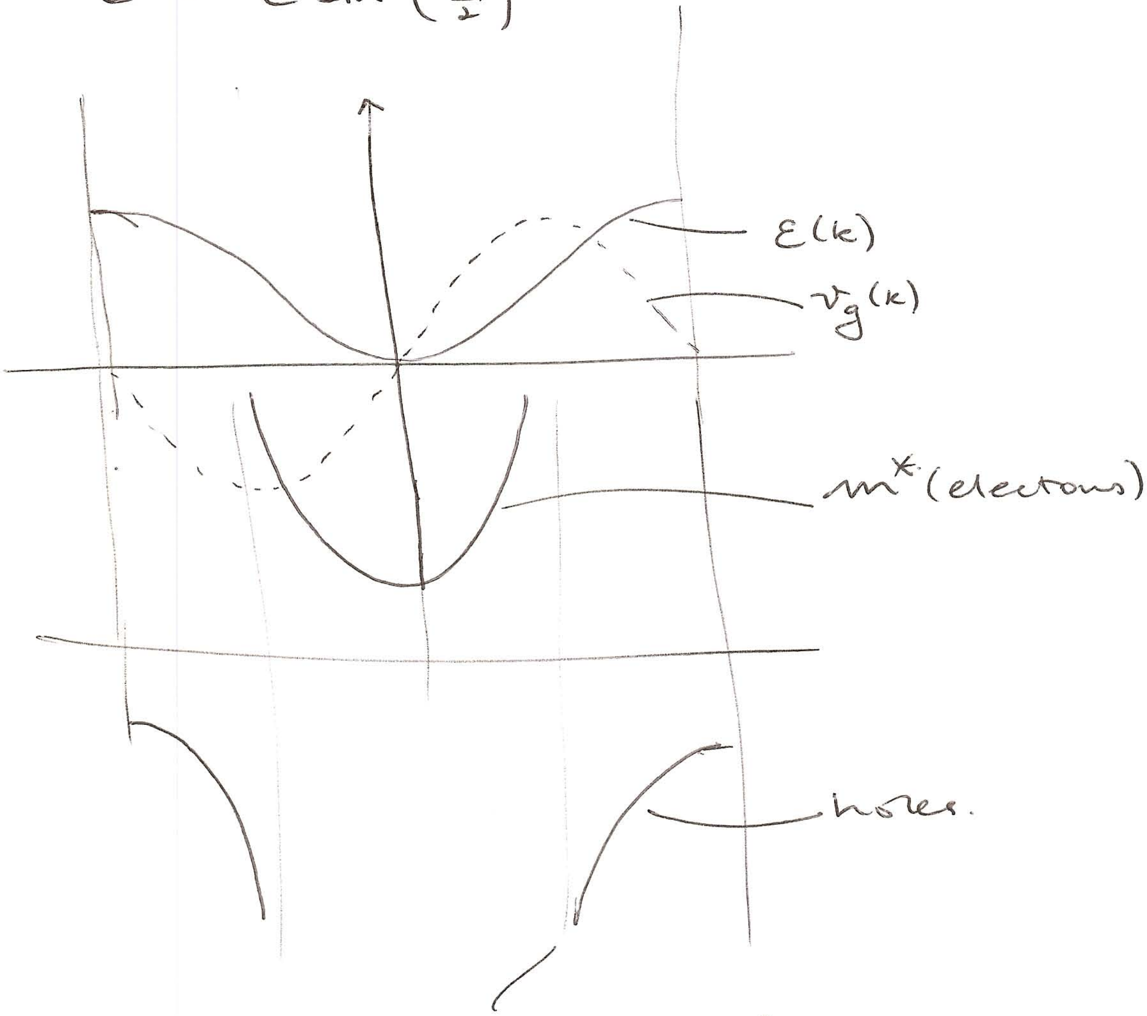
- Specific heat of a metal $\propto kT$

— good thermal & electrical conductivity

- Pauli paramagnetism (see chapter 10).

Question 24

$$\mathcal{E} = C\sin^2\left(\frac{ka}{2}\right)$$



note that $\frac{1}{m^*} \propto \frac{d^2\mathcal{E}}{dk^2}$

— so $m^* \to \infty$ at points of inflection.

$$-eE = \hbar\dot{k} \Rightarrow k(t) = k(0) - \frac{eEt}{\hbar}$$

$$v_g = \frac{1}{\hbar}\frac{\partial\mathcal{E}}{\partial k} = \frac{Ca}{2\hbar}\sin\left[a\left(k(0) - \frac{eEt}{\hbar}\right)\right]$$

$$x(t) = \frac{C}{2eE}\cos\left[a\left(k(0) - \frac{eEt}{\hbar}\right)\right]$$

— For data in fig 9.2

$$E = V/L \qquad L \simeq 1\mu m$$

$$\nu_{Bloch} = \frac{eV}{h} \cdot \frac{a}{L}.$$

fit the slope. (offsets not controlled)

$$a = L\left(\frac{h}{e}\right)\frac{d\nu_B}{dV} \simeq 15\,nm.$$

For. GaAs.

$$\omega_{Bloch} = \frac{eEa}{\hbar} \qquad a \sim 0.2\,nm \quad \boxed{E} \sim 100\ Vm^{-1}$$

$$\Rightarrow \omega_{Bloch} = 3 \times 10^7.\ s^{-1}$$

$$\nu_{Bloch} \simeq 2 \times 10^8\ Hz.$$

$$\text{Amplitude} = \frac{\text{bandwith}}{eE} \sim \frac{10\,eV}{100\,eV}.\,m.$$

$$\sim 10^{-1}\,m$$

Much larger than any reasonable mean free path $\Rightarrow$ electron scatters and will not conserve momentum $\Rightarrow$ not observable.

Question 25

11.23

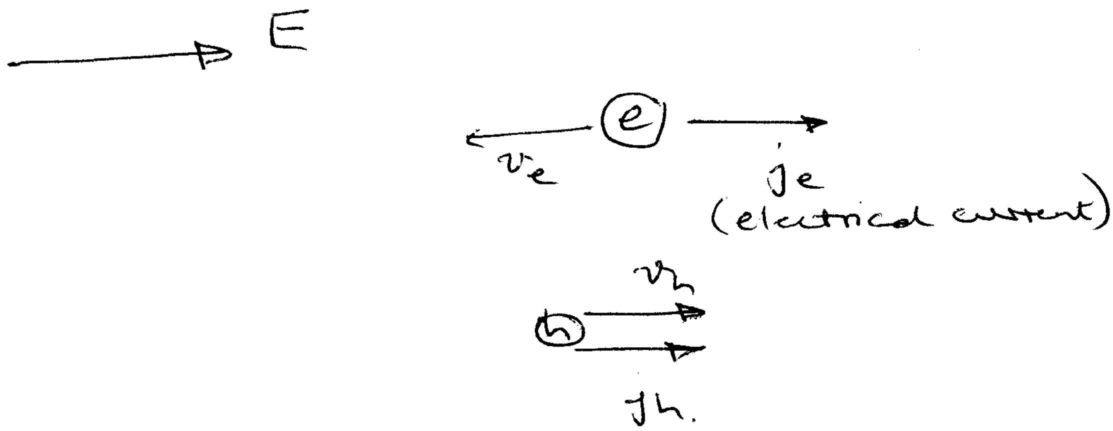
Electron: $\hbar \dot{k}_e = -eE$

Hole : $\hbar \dot{k}_h = eE$.

Note that although $k_h = -k_e$

$$\mathcal{E}_h(k_h) = -\mathcal{E}_e(k_e)$$

Thus $v_h = \dfrac{\partial \mathcal{E}_h}{\partial k_h} = \dfrac{\partial \mathcal{E}_e}{\partial k_e} = v_e$



$\Rightarrow$ Same direction of electrical current

— Opposite velocity

Question 26

Cyclotron resonance

$$E(k) = \frac{1}{2}\frac{\hbar^2 k^2}{m^*}$$

$$\vec{v} = \frac{1}{\hbar}\vec{\nabla}_k E(k) = \frac{\hbar \vec{k}}{m^*}$$

Eqn of motion

$$\hbar\dot{\vec{k}} = -\frac{e\hbar}{m^*}\vec{k}\times\vec{B}$$

Put $B = B\hat{z} \Rightarrow \dot{k}_z = 0.$

$$\frac{d\vec{k}}{dt} = \frac{eB}{m^*}\hat{z}\times\vec{k}$$

$\Rightarrow$ Circular motion with angular velocity $\omega_c = \frac{eB}{m^*}$

For a hole $e \to -e$

Sense of orbit opposite to that for electron

Equ. 11.27 can ~~be~~ written $\left(\beta = \frac{eB\tau}{m} = \omega_c\tau\right)$

$$v_x + \beta v_y = -\frac{e\tau}{m} E_x$$

$$-\beta v_x + v_y = -\frac{e\tau}{m} E_y$$

$$v_z = -\frac{e\tau}{m} E_z$$

$$\begin{bmatrix} 1 & \beta & 0 \\ -\beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = -\frac{e\tau}{m} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Note that $\begin{pmatrix} 1 & -\beta & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1+\beta^2 \end{pmatrix} \begin{pmatrix} 1 & \beta & 0 \\ -\beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1+\beta^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\therefore \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = -\frac{e\tau}{m} \cdot \frac{1}{1+\beta^2} \begin{bmatrix} 1 & -\beta & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1+\beta^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\vec{J} = -ne\vec{v}$$

$$\therefore \sigma = \frac{ne^2\tau}{m} \cdot \frac{1}{1+\beta^2} \begin{bmatrix} 1 & -\beta & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1+\beta^2 \end{bmatrix}$$

NB. σ_{zz} indep. of field.