NATURAL SCIENCES TRIPOS Part II

Wednesday 17 January 2018 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **all** *questions to the best of your abilities. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 The action for a system consisting of a relativistic charged particle moving in an electromagnetic field is given by

$$S = -\int mc^2 \, \mathrm{d}\tau - \int eA_\mu \, \mathrm{d}x^\mu(t),$$

where $x^\mu = (ct, \boldsymbol{x})$, $A^\mu = (\phi/c, \boldsymbol{A})$, and τ is the proper time.

(a) Derive the equations of motion in terms of the electric and magnetic fields, given by $\boldsymbol{E} = -\nabla\phi - \frac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, respectively. [8]

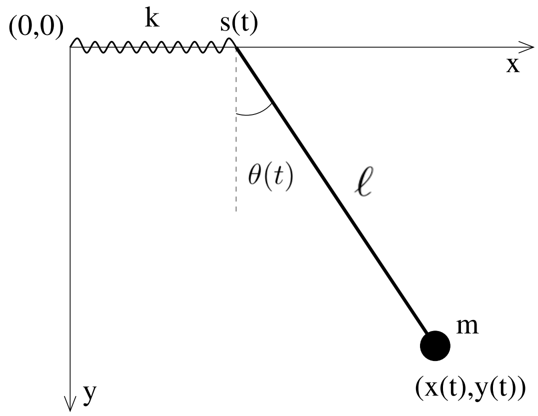
(b) Suppose that $\boldsymbol{B} = 0$, that $\boldsymbol{E}$ is constant and that at $t = 0$ the particle has velocity $\boldsymbol{v}_0$. Find the subsequent velocity of the particle. [5]

(c) Find the limiting velocity of the particle as $t \to \infty$. [3]

(d) Suppose that instead $\boldsymbol{E} = 0$ (and generically $\boldsymbol{B} \neq 0$). Show that γ, and hence the total speed, are constant. [4]

(e) Suppose now that $\boldsymbol{E} = 0$ and $\boldsymbol{B}$ is constant. Show that the time dependence of the perpendicular velocity vector $\boldsymbol{v}_\perp = \boldsymbol{v} - \boldsymbol{B}\frac{(\boldsymbol{v}\cdot\boldsymbol{B})}{B^2}$ is periodic and find the period. [5]

2 A massless rod of length ℓ makes an angle $\theta(t)$ with the vertical, has a point mass m at one end, and is in a constant gravitational field $\boldsymbol{g} = g\hat{\boldsymbol{y}}$. The other end of the rod is attached to a horizontal line with a frictionless hinge, and connected to a point along the line by a massless spring of constant k and zero rest length, as illustrated in the figure.



Let us call $s(t)$ the instantaneous horizontal displacement of the hinge from the origin (i.e., the fixed point of the spring).

(a) Show that the Lagrangian for this system can be written in terms of the dimensionless variables $\eta(t) = s(t)/\ell$ and $\theta(t)$ as

$$L = \frac{1}{2}\left[\dot{\eta}^2 + 2\dot{\eta}\dot{\theta}\cos\theta + \dot{\theta}^2\right] + \frac{g}{\ell}\cos\theta - \frac{1}{2}\frac{k}{m}\eta^2\,,$$

up to a multiplicative constant. [10]

(b) Find the equations of motion for this system. Then assume that the dynamical variables and their derivatives are small and show that the equations of motion expanded to linear order can be written as

$$\begin{cases} \ddot{\theta} + \ddot{\eta} + \omega_0^2\theta = 0 \\ \ddot{\theta} + \ddot{\eta} + \omega_1^2\eta = 0\,. \end{cases}$$

Find the values of the characteristic frequencies ω_0 and ω_1. [8]

(c) Assuming a solution of the form $\theta(t) = \theta_0 \sin(\omega t)$, find the corresponding form for $\eta(t)$ and the allowed values of ω. Consider the limit $k \to \infty$ and discuss briefly whether the result is consistent with your expectation or not. [7]

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3 (a) Explain why a total derivative term in the Lagrangian (or Lagrangian density) of a dynamical system does not affect the equations of motion and may be discarded. [4]

(b) A system is described by a real scalar field $h(\boldsymbol{x}, t)$ with a Lagrangian density containing spacetime derivatives of $h(\boldsymbol{x}, t)$ up to and including second order, $\mathcal{L}(h, \partial_\mu h, \partial_\mu \partial_\nu h)$. Derive the corresponding Euler-Lagrange equations of motion. [10]

(c) The height $h(\boldsymbol{x}, t)$ of a surface grown over the $\boldsymbol{x} = (x^1, x^2)$ plane by random deposition of atoms is described by the action

$$S = \int d^2\boldsymbol{x}\, dt \left(\frac{\partial h}{\partial t} - \nu \nabla^2 h \right)^2 ,$$

where ν is a positive constant. Find the Euler–Lagrange equation of motion governing the dynamics of $h(\boldsymbol{x}, t)$. [7]

(d) What symmetries does the system possess? [4]

4 Consider the Lagrangian density of a 1-dimensional elastic rod with density $\rho = 1$ and elastic constant $\kappa = 1$,

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 ,$$

where $\phi(x,t)$ is the local displacement field.

(a) Obtain the Euler-Lagrange equations of motion for the field. [3]

(b) Obtain the total angular momentum tensor of the system. Is it conserved? Justify your answer and comment briefly on its significance in relation to the symmetries of the system. [7]

(c) Consider adding a viscous damping term to the equation of motion of the rod, $\gamma\partial_t\partial_x^2\phi$ where γ is a positive constant. Obtain the Fourier transform $\hat{G}(k,\omega)$ of the corresponding Green's function $G(x,t)$, which solves the equation of motion with a delta function driving in space and time: [5]

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \gamma\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\right) G(x,t) = \delta(x)\delta(t) .$$

[*Use the Fourier transform conventions:*

$$\tilde{G}(k,t) = \int \hat{G}(k,\omega)\, e^{-i\omega t}\, \frac{\mathrm{d}\omega}{2\pi} \qquad\qquad G(x,t) = \iint \hat{G}(k,\omega)\, e^{-ikx-i\omega t}\, \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2} .$$
]

(d) Assuming that $k^2 < 4/\gamma^2$, obtain an expression for $\tilde{G}(k,t)$ by Cauchy integration. Then take the limit $\gamma \to 0$ and compute $G(x,t)$. Is the result consistent with the choice of initial conditions and with $\rho = \kappa$? [10]

[*You may need the integral:*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx}\, \mathrm{d}s = \mathrm{TH}(x) ,$$

where $\mathrm{TH}(x)$ *is the top hat function, which takes value* 1 *for* $-1 < x < 1$ *and vanishes otherwise.*]

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