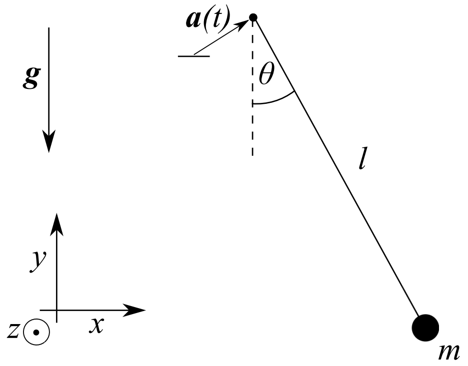
NATURAL SCIENCES TRIPOS   Part II

Wednesday 14 January 2015   10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A massless rod of length l makes an angle $\theta(t)$ with the vertical, has a mass m at one end, and is in a constant gravitational field $\boldsymbol{g} = -g\hat{\boldsymbol{y}}$. The other end of the rod is attached with a frictionless hinge to a support, so the rod can rotate in the xy-plane. The support itself also moves in the xy-plane, having a changing displacement $\boldsymbol{a}(t) = (a_x(t), a_y(t), 0)$ from a fixed origin.



(a) Show that the Lagrangian for this system is: [8]

$$L = \frac{1}{2}m[\dot{a}_x^2 + \dot{a}_y^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}(\dot{a}_x\cos\theta + \dot{a}_y\sin\theta)] - mg(a_y - l\cos\theta).$$

(b) Show that the equation of motion for θ can be written in the form:

$$l^2\ddot{\theta}\hat{\boldsymbol{z}} = \boldsymbol{l} \times (\boldsymbol{g} - \ddot{\boldsymbol{a}}),$$

where the vector $\boldsymbol{l}$ has length l and points from the hinge to the mass. [8]

(c) The support executes small but rapid horizontal oscillations, $\boldsymbol{a}(t) = a\cos(\omega t)\hat{\boldsymbol{x}}$, with $a/l \ll 1$ and $\omega \gg \sqrt{g/l}$. In this limit the pendulum's motion is well represented by

$$\theta = \theta_1 - \frac{a}{l}\cos(\theta_1)\cos(\omega t),$$

(TURN OVER for continuation of question 1

where θ_1 is a large slowly varying angle and the second term is a small but fast oscillation at ω. Use this separation of time-scales and amplitudes to find an effective equation of motion for $\theta_1(t)$ by expanding the complete equation of motion around $\theta_1(t)$ to first order in a/l, and then averaging over the time period of one fast oscillation. [9]

(d) Show that the variation of θ_1 would be reproduced by a simple pendulum (without a moving support) with potential energy given by [8]

$$V_{\text{eff}}(\theta) = ml\left(-g\cos(\theta) + \frac{a^2\omega^2}{4l}\cos^2(\theta)\right),$$

and hence calculate how large a must be for the $\theta = 0$ configuration to lose stability.

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2 A non-conducting circular elastic ring has natural length $2\pi r_0$, spring constant k, mass m, and carries an electric charge q which is distributed evenly along its length. The ring is constrained to sit in the xy-plane and a perpendicular constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ is applied through the ring. The elastic ring spins around its axis (i.e. the z axis) and its radius starts to increase, but the ring always remains circular and centred at the origin.

(a) The elastic ring is spinning at $\dot{\phi}$ and has instantaneous radius r. In the gauge where $\boldsymbol{A} = (Br/2)\hat{\boldsymbol{\phi}}$ and neglecting the electric self-interaction of the ring, show the Lagrangian for this system is: [7]

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - 2\pi^2 k(r - r_0)^2 + qB\frac{r^2}{2}\dot{\phi}.$$

(b) Show that the Hamiltonian for this system has the value

$$H = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + 2\pi^2 k(r - r_0)^2.$$

Notice that B is absent from this expression. Explain why the magnetic field will nevertheless appear in Hamilton's equations of motion. [5]

(c) Identify two symmetries of this system (in addition to the gauge symmetry of the magnetic field) and their corresponding conserved quantities. Using one of them, show the Euler-Lagrange equations of motion can be reduced to:

$$m\ddot{r} = \frac{J^2}{mr^3} - \frac{q^2B^2r}{4m} - 4\pi^2 k(r - r_0),$$

where J is a constant of the motion. [8]

(d) The elastic ring is released with $r = r_0$, $\phi = 0$, $\dot{\phi} = -qB/(2m)$ and $\dot{r} = 0$. Show that this choice of initial conditions corresponds to $J = 0$ and that the subsequent motion of the radius is of the form

$$r = r_e + (r_0 - r_e)\cos(\omega t).$$

Find expressions for r_e and ω. Find the motion of ϕ. [6]

(e) The ring is stretched to radius $r = r_1$ and released from rest. If the natural radius of the elastic ring, r_0, is negligible, what is the minimum radius the ring will reach in its subsequent motion? What is its radial and angular velocity when it reaches this radius? [7]

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3 (a) Define Poisson brackets and discuss their use in classical mechanics: (i) to define canonical transformations; and (ii) to write the equation of motion of a generic observable $\mathcal{O}(q, p)$. [6]

(b) Consider the following Hamiltonian of a particle of canonical coordinate q and momentum p:

$$H = p \cosh(2q).$$

Compute Hamilton's equations of motion. Using the Poisson brackets, derive the equation of motion of $\mathcal{O}(q, p) = e^{qp}$. [6]

(c) Consider the change of variables

$$Q = f(p) \sinh(q) \qquad \text{and} \qquad P = f(p) \cosh(q).$$

What equation must $f(p)$ satisfy to be a canonical transformation? With the help of trigonometric properties of the hyperbolic sine and cosine functions, choose $f(p)$ so that the transformation is canonical *and* the Hamiltonian can be written as:

$$H = \frac{Q^2}{2} + \frac{P^2}{2}.$$

in terms of Q and P. (You may assume $p > 0$ in this part of the question.) [9]

(d) Consider applying a time-dependent perturbation $g(t)$ conjugate to the dynamical variable $Q(t)$ by changing the Hamiltonian to $H - g(t)Q(t)$. Using Hamilton's equations of motion in the new variables Q and P of the perturbed system, obtain the second order differential equation for Q alone. Then use the Green's function method to obtain the solution $Q(t)$ for the specific case where the driving term takes the form

$$g(t) = \frac{\omega_0}{2} \exp\left(-\omega_0 |t|\right) \qquad (\omega_0 > 0).$$

Poles on the real axis should be moved according to the principle of causality.

[*Hint: It may be convenient to use the formula:*

$$Q(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, G(\omega) g(\omega) e^{-i\omega t}$$

where $G(\omega)$ is the Fourier transformed Green's function and $g(\omega) = \omega_0^2/(\omega^2+\omega_0^2)$ is the Fourier transform of the driving force above.]

[12]

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4 A very long elastic string with mass per unit length ρ is aligned along the x axis and an outwards force F pulls at both ends so the string carries a tension F.

(a) The string undergoes y and z displacements ψ_y and ψ_z. Show that, if the spatial derivatives of these displacements are small, the Lagrangian for the string is [7]

$$L = \int \frac{1}{2}\rho \left[\left(\frac{\partial \psi_y}{\partial t}\right)^2 + \left(\frac{\partial \psi_z}{\partial t}\right)^2 \right] - \frac{1}{2}F \left[\left(\frac{\partial \psi_y}{\partial x}\right)^2 + \left(\frac{\partial \psi_z}{\partial x}\right)^2 \right] \mathrm{d}x.$$

(b) Show the Lagrangian is invariant under the transformation $\psi_y \to \psi_y + a$, where a is a constant displacement. Find the corresponding conserved quantity, show that it is conserved, and explain its physical significance. [5]

(c) Show the Lagrangian is also invariant under a global rotation of the string's displacement around the x axis. Find the corresponding conserved quantity and explain its physical significance. [7]

(d) The string in tension is now replaced by an elastic rod in compression, so F is negative, and we restrict attention to displacements in the y direction, setting ψ_z and all its derivatives to zero. In this case we must add another term to the potential energy,

$$\int \frac{1}{2}B \left(\frac{\partial^2 \psi_y}{\partial x^2}\right)^2 \mathrm{d}x,$$

that penalizes bending of the rod. Show that waves on the rod will obey the dispersion relation: [7]

$$\omega^2 = \frac{k^2}{\rho}(F + Bk^2).$$

[*Hint: To deal with the second derivative in L, repeat the regular derivation of the Euler Lagrange equations but integrate by parts twice.*]

(e) If the rod has length l and $\psi_y = 0$ at its ends show, using the above dispersion relation or otherwise, that the straight compressed rod is unstable if $l > \pi\sqrt{-B/F}$. [7]

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5 (a) In the lecture notes, you have studied a way to couple complex scalar fields to electromagnetism, which leads to a Lagrangian density that is invariant under local phase change. Discuss briefly how this is done by introducing the concept of covariant derivative; write an explicit expression for the resulting Lagrangian and demonstrate that it is invariant under local phase change provided that it is accompanied by an appropriate simultaneous gauge transformation. [6]

(b) Discuss briefly Noether's theorem and how it relates symmetries and conservation laws. You should include a general form of the conserved current and conserved charge. [5]

(c) Consider the following Lagrangian density:

$$\mathcal{L} = (\partial_\mu \phi^*)\,(\partial^\mu \phi) - m^2 \phi^* \phi + ieA_\mu(\boldsymbol{r}, t)\,[\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi]\,.$$

Find the Euler-Lagrange equations and obtain the Hamiltonian density $\mathcal{H}$ expressed as a function of its proper variables. [12]

(d) The Lagrangian density $\mathcal{L}$ in part (c) is invariant under global phase change $\phi \to e^{-i\epsilon}\phi$ and $\phi^* \to e^{i\epsilon}\phi^*$. Use the result of Noether's theorem to compute the conserved current J^μ. Then demonstrate explicitly that $\partial_\mu J^\mu = 0$ using the Euler-Lagrange equations obtained earlier. [10]

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6 (a) Explain briefly the principles of *mean field theory* and under what circumstances it is expected to be a reliable approximation. [5]

(b) Consider a large 2D square lattice of spins $\boldsymbol{S}_i$, $i = 1, \ldots, N$, with unit norm $|\boldsymbol{S}_i| = 1$ and energy

$$E = -\frac{J}{2} \sum_{i,\delta} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+\delta} \qquad (J > 0).$$

The summation is over all sites i and their respective four nearest neighbouring sites labelled by $i + \delta$ (ignore boundary effects.). Using mean field theory (MFT), compute the approximate partition function of the system, $Z = \sum_{\{\boldsymbol{S}_i\}} \exp(-\beta E)$. Notice that the average value of a spin is a vector $\boldsymbol{S} = \langle \boldsymbol{S}_i \rangle$ whose norm S can be smaller than 1. It is convenient in the calculations to adopt a reference frame for the spins $\boldsymbol{S}_i$ such that $\boldsymbol{S}$ points along the z axis. [7]

[*Hint: using polar coordinates, one can express the summation over a single spin at site i as*

$$\sum_{\boldsymbol{S}_i} \exp\left(\frac{\beta J}{2} \boldsymbol{S}_i \cdot \boldsymbol{S}\right) = 2\pi \int_0^\pi \sin\theta_i \, \exp\left(\frac{\beta J}{2} S \cos\theta_i\right) \, \mathrm{d}\theta_i.$$
]

(c) Within MFT, compute the expectation value of the z-component of a spin

$$\langle \boldsymbol{S}_k \cdot \hat{z} \rangle = \frac{1}{Z} \sum_{\{\boldsymbol{S}_i\}} (\boldsymbol{S}_k \cdot \hat{z}) \exp(-\beta E)$$

and derive from it the self-consistency condition for the z-component of the average spin $\boldsymbol{S}$. Show that it can be expressed in the form $\tau x = \coth(x) - 1/x$, where $x \propto S$. Find the expressions for x and for the proportionality constant τ. [11]

(d) Discuss how the (graphical) solutions to the self-consistency equation vary as a function of τ and find the transition temperature T_c in terms of the parameters of the system. Comment on the physical significance of the result. [*Hint: you should think carefully about the behaviour of the function* $\coth(x) - 1/x$ *near the origin.*] [5]

(e) The order parameter for the model discussed here is a 3-component vector $\boldsymbol{m}(x)$. Assume the following free energy:

$$f = f_0 + a(T - T_c)\, \boldsymbol{m}(x) \cdot \boldsymbol{m}(x) + b\, [\boldsymbol{m}(x) \cdot \boldsymbol{m}(x)]^2 + c\, [\nabla \boldsymbol{m}(x)] \cdot [\nabla \boldsymbol{m}(x)],$$

with a, b, c positive constants. Find the value(s) of the field $\boldsymbol{m}(x)$ that minimise the free energy as a function of T. What type of transition takes place at $T = T_c$? What symmetry is spontaneously broken? [5]

END OF PAPER