NATURAL SCIENCES TRIPOS Part II

Wednesday 17 January 2007 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A dynamical system has position co-ordinates q_i and canonical momenta p_i. Write the Hamiltonian $H(q_i, p_i, t)$ in terms of the system Lagrangian $L(q_i, \dot{q}_i, t)$ and these quantities. A non-relativistic particle of mass m and charge q moves in an electromagnetic field produced by an electrostatic potential ϕ and magnetic vector potential $\mathbf{A}$. Show that the Hamiltonian is

$$H = \frac{|\mathbf{p} - q\mathbf{A}|^2}{2m} + q\phi \ .$$ [8]

In Cartesian coordinates (x, y, z) the electric field is $\mathbf{E} = (0, -m\omega_0^2 y, 0)$ and the magnetic field is $\mathbf{B} = (0, 0, B)$. Show that $\phi = m\omega_0^2 y^2/2$, $\mathbf{A} = (-By, 0, 0)$ are suitable choices for the potentials. [4]

For a particle moving in this field, show that the momenta p_x, p_z and the Hamiltonian H are constants of the motion. [4]

Find Hamilton's equations of motion for the variables p_y, x, y and z and show that

$$\ddot{y} + (\omega^2 + \omega_0^2)y = \frac{p_x \omega}{m},$$

where $\omega \equiv qB/m$. [10]

Hence find the general solutions for $x(t), y(t)$ and describe the motion in the x, y plane of a particle initially moving with velocity $v = (v_x, 0, 0)$. [8]

2 Describe briefly how the principle of least action leads to Lagrange's equations of motion for a dynamical system having coordinates and velocities $(q_i, \dot{q}_i)$. [6]

A mechanical governor used to control the speed of a steam engine consists of the configuration shown in the figure:

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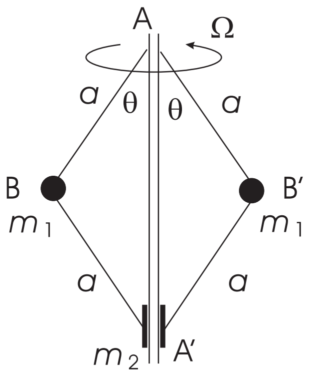

(i) the vertical axis AA' rotates at a constant angular velocity Ω;
(ii) light rods $AB, AB', A'B, A'B'$ each of length a are freely pivoted at A, B, A', B';
(iii) the pivot at A is fixed, so that the pivot at A' moves as the angle θ changes;
(iv) masses m_1 are attached at B and B' and a mass m_2 is free to slide on the vertical axis at A'.

Show that the Lagrangian of the system is given by

$$L = m_1 a^2(\Omega^2 \sin^2\theta + \dot{\theta}^2) + 2m_2 a^2 \dot{\theta}^2 \sin^2\theta + 2ag\cos\theta\,(m_1 + m_2)\ . \qquad [6]$$

Find the equation of motion of the system. [7]

Show that the system can rotate in equilibrium with $\theta = 0$ unless Ω exceeds a certain critical velocity. Determine the equilibrium angle θ_0 for the case when Ω is greater than this critical value. [8]

Show that the angular frequency of small oscillations about the equilibrium angle θ_0 is given by $\Omega\sin\theta_0/\sqrt{1 + 2(m_2/m_1)\sin^2\theta_0}$. [7]

3 Consider the following generalisation of the Lagrangian for a simple relativistic particle:

$$L = -m_0\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\ ,$$

where $(dx^0, dx^1, dx^2, dx^3) = (cdt, dx, dy, dz)$, and $g_{\mu\nu}$ is a symmetric tensor which varies with position and time, and m_0 is a constant.

For the simplifying case of only time plus one dimension of space, with $g_{00} = -g_{11} = g(x)$, independent of time, and also $g_{10} = g_{01} = 0$, show that the Euler-Lagrange equations reduce to the form [8]

$$\frac{d}{dt}(\Gamma m_0 v) = -\frac{m_0}{\Gamma}\frac{\partial\phi}{\partial x}$$

and give an explicit expression for the function Γ and the potential ϕ in terms of $g(x)$ and $v = dx/dt$. [8]

For the general case show that the equations of motion are given by

$$\frac{d}{dt}\left(\gamma g_{k\nu}\frac{dx^\nu}{dt}\right) = \frac{1}{2}\gamma\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{\partial g_{\mu\nu}}{\partial x^k}$$

giving the explicit expression for γ and indicating carefully what values are taken by the indices k, μ, ν. [10] [8]

4 The three-dimensional Fourier transform of an electric charge density distribution $\rho(\mathbf{r})$ can be written as

$$\tilde{\rho}(\mathbf{k}) \equiv \int \mathrm{d}^3\mathbf{r}\ \rho(\mathbf{r}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \ .$$

Write down the formula for the inverse Fourier transform. [4]

If it is placed within a dielectric medium with dielectric constant ϵ_0 the associated electrostatic potential $\varphi(\mathbf{r})$ is determined by the Poisson equation

$$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} \ .$$

Find the relationship between the Fourier transforms $\tilde{\rho}(\mathbf{k})$ and $\tilde{\varphi}(\mathbf{k})$. Explain how the potential can be found in terms of an integral over $\mathbf{k}$ if the charge density is known. [8]

A uniform metallic layer occupying the region $-t \le z \le t$ and extending infinitely in the x, y plane is embedded in a dielectric medium with $\epsilon_0 = 1$. A charge density wave $\rho(\mathbf{r}) = A\cos(Qx)$ is set up in the layer by perturbing the electron distribution. Calculate the Fourier transform $\tilde{\rho}(\mathbf{k})$, where $\mathbf{k} \equiv (k_x, k_y, k_z)$. [10]

Calculate the potential at the point $(x, 0, 0)$, expressing the answer in terms of $I(a)$, where

$$I(a) \equiv \int_{-\infty}^{\infty} \mathrm{d}k\ \frac{\sin k}{(a^2+k^2)k} \ .$$ [5]

By using a contour integral, show that

$$I(a) = \frac{\pi}{a^2}\left(1 - \exp(-a)\right) \ .$$ [7]

5 Describe how the Cauchy integral theorem can be used to evaluate contour integrals in the complex plane. Illustrate your answer by showing that

$$\int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{2}$$

(*Hint:* you may find it useful to note that the integrand has a pole at $x \to \infty$ and a branch cut between $x = -1..1$) [17]

The Cauchy integral theorem can be used to evaluate infinite series. By considering the identity

$$\oint_C \frac{\pi \operatorname{cotan} \pi z}{(a+z)^2}\mathrm{d}z = 0,$$

where the contour C is a circle of infinite radius in the complex plane centered about $z = 0$, show that [17]

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

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6 Discuss the concept of discrete transition probability $w(k, k')$ for a discrete one-dimensional random walk. [6]

Consider an ensemble of small identical spherical Brownian particles, of radius a and density ρ, suspended in a container filled with water (density ρ_0). Derive the modified diffusion equation for the probability $P(z, t)$ of finding a particle at a height z taking into account only first-order corrections in powers of $\tilde{m}ga/k_{\rm B}T$ (assumed small): [12]

$$\frac{\partial P}{\partial t} = \frac{1}{2} D \left(\frac{\partial^2 P}{\partial z^2} + \frac{\tilde{m}g}{k_{\rm B}T} \frac{\partial P}{\partial z} \right)$$

where $\tilde{m} = 4\pi(\rho - \rho_0)a^3/3$.

Derive the equilibrium Boltzmann distribution of these particles along the vertical z-axis. [6]

For $\rho = 1.1 \times 10^3$ kg m^{-3} and $\rho_0 = 1 \times 10^3$ kg m^{-3} estimate the order of magnitude of the radius a of a particle for which the effect of Brownian diffusion is relevant, such that the trajectory of moving particle deviates significantly from a straight line. [10]

[At room temperature $k_{\rm B}T \sim 4 \times 10^{-21}$ J.]