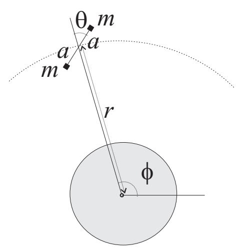
NATURAL SCIENCES TRIPOS Part II

Wednesday 18 January 2006 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **3** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Two equal masses m joined by a length $2a$ of light thin wire are in orbit about a planet of mass M. The wire is in the plane of the orbit at all times. Write down a Lagrangian for the system in terms of the polar coordinates (r, ϕ) of its midpoint and the angle θ between the wire and the line to the planet centre, assuming $a \ll$ distance from the planet and the gravitational attraction between the two little masses is negligible. [7]



Find the equation of motion for θ. [7]

Determine the stability of equilibrium values of θ. [8]

If the orbit is circular and the system is close to a stable value of θ, how many oscillations does it perform in one orbital period? [12]

2 Describe the terms in the Bernoulli equation in fluid dynamics, and the conditions when it applies. [10]

Consider a bath of cross-sectional area $0.6\,\mathrm{m}^2$, filled to a depth of $20\,\mathrm{cm}$. Estimate how long it takes to empty when the plug is removed, if the area of the plug hole is $10\,\mathrm{cm}^2$. [12]

Two cylindrical jets of water, which have the same radius a and velocity components $(0, 0, v)$ and $(0, 0, -v)$ respectively, meet head-on at the origin and spread to form a sheet in the $z = 0$ plane. Show the thickness of this sheet at distance r from the origin is a^2/r. [12]

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3 The Lagrangian L of a particle of mass m and charge e moving with velocity $\boldsymbol{v}$ in an electrostatic potential ϕ is $\frac{1}{2}mv^2 - e\phi$. Using the requirement that $\int L\,dt$ should be Lorentz invariant, or otherwise, explain why the generalisation of the potential energy term in the Lagrangian $L = T - V$ to $V = e(\phi - \boldsymbol{v}\cdot\boldsymbol{A})$ is required, where $(\phi, \boldsymbol{A})$ is the electromagnetic 4-potential. [6]

Consider the non-relativistic motion of the charged particle in a constant magnetic field $\boldsymbol{B}$ which is directed along z-axis. In cylindrical polar coordinates (r, θ, z):

(a) By using the Stokes theorem for $\int \boldsymbol{A}\cdot d\boldsymbol{l}$, or otherwise, show that $\{A_\theta = \frac{1}{2}Br,\ A_r = 0,\ A_z = 0\}$ represents a magnetic field with a z-component B. [8]

(b) Write down the Lagrangian and derive the equations of motion: in particular, show that [8]

$$\dot{\theta} = -\frac{eB}{2m} - \frac{J}{r^2}$$

where J is a constant.

(c) Show that the radius of the helical orbit must be proportional to $B^{-1/2}$ and the angular frequency of particle on this orbit is $|\dot{\theta}| = eB/m$. [8]

(d) Further show that the helical pitch angle ψ (the angle between $\boldsymbol{B}$ and $\boldsymbol{v}$) obeys $\tan\psi \propto B^{1/2}$. [4]

4 Describe how contour integration methods can be used to evaluate definite integrals of functions with no poles. Illustrate your answer by showing that:

$$\int_0^\infty \mathrm{d}x \cos(x^2) = \int_0^\infty \mathrm{d}x \sin(x^2) = \frac{\sqrt{2\pi}}{4}$$ [10]

(Hint: use a wedge-shaped contour with angle $\pi/4$, not forgetting that $e^{i\pi/4} = (1+i)/\sqrt{2}$.)

How are poles on the real axis treated in contour integration? Illustrate your answer by showing that:

$$\int_{-\infty}^\infty \mathrm{d}x\,\frac{\sin(x)}{x} = \pi$$ [12]

(Hint: $\sin z = (e^{iz} - e^{-iz})/2i$)

How are branch cuts dealt with in contour integration? Illustrate your answer by showing that:

$$\int_0^\infty \mathrm{d}x\,\frac{x^\alpha}{1+\sqrt{2}x+x^2} = \sqrt{2}\pi\frac{\sin(\alpha\pi/4)}{\sin(\alpha\pi)}\ .$$ [12]

for $-1 < \alpha < 1$.

5 Derive the Kramers-Kronig relations between the real and imaginary parts of the generalized susceptibility for a perturbation potential of the form $V = -x.f$ where $x(t)$ is a position coordinate and $f(t)$ a force. [8]

The equation of motion for a damped harmonic oscillator has the form:

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = f(t)$$

Derive the relationship between the Green's function for this differential equation $G(t-t')$, the position coordinate x and the force f. [4]

Derive an expression for the Fourier Transform of the Green's function $G(\omega)$ and write down the Kramers-Kronig relations for its real and imaginary parts. [7]

Convert the principal-value integrals to contour integrals and find their poles. [7]

By evaluating one of these integrals show that the corresponding Kramers-Kronig relation is obeyed by this Green's function. [8]

(Hint: $\mathsf{P}\int_{-\infty}^{\infty} f(x)dx = \mathrm{Lim}_{\epsilon\to 0}[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}]f(x)dx$ for a pole at $x = 0$.)

6 Thermal (Johnson) noise in a resistor is modelled as a white-noise voltage source, with a mean-square amplitude $\langle V^2\rangle = 2Rk_BT$, connected in series with an ideal resistor R. Consider a circuit consisting of such a resistor connected across a capacitor C. Show that the equation of motion for the charge on the capacitor obeys a Langevin equation. [6]

Describe the standard form for the Langevin (stochastic) equation,

$$\dot{q}_\alpha = F_\alpha(\boldsymbol{q}) + G_\alpha^k(\boldsymbol{q})A_k(t) \ ,$$

and its relation to classical Brownian motion. [6]

Outline principles leading to the Fokker-Planck (kinetic) equation for the probability density $f(\boldsymbol{q}, t)$: [8]

$$\frac{\partial f(\boldsymbol{q},t)}{\partial t} = \left\{ -\frac{\partial K_\alpha(\boldsymbol{q})}{\partial q_\alpha} + \frac{1}{2}\frac{\partial^2}{\partial q_\alpha \partial q_\beta} Q_{\alpha\beta}(\boldsymbol{q}) \right\} f(\boldsymbol{q},t) \ ,$$

$$\text{with} \qquad K_\alpha = F_\alpha + \frac{1}{2}\frac{\partial G_\alpha^i}{\partial q_\beta} G_\beta^k \delta_{ik}; \quad Q_{\alpha\beta} = G_\alpha^i G_\beta^k \delta_{ik} \ .$$

Obtain the Fokker-Planck equation for the probability density $f(q,t)$ of the charge on the capacitor. [4]

Show that the equilibrium probability density $f(q)$ of the charge on the capacitor is proportional to $\exp[-q^2/2Ck_BT]$ and comment on its form. [10]