NATURAL SCIENCES TRIPOS Part II

Wednesday 16 January 2002 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A dynamical system has Lagrangian $L(q_i, \dot{q}_i, t)$. Define the canonical momenta p_i and the Hamiltonian $H(q_i, p_i, t)$. Use Hamilton's equations to show that for any scalar function $f(q_i, p_, t)$ the evolution equation can be written

$$\frac{\mathrm{df}}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

where the Poisson bracket of two functions $\{f, g\}$ is defined as

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$ [7]

For a vector function $\boldsymbol{A}(\boldsymbol{x})$ of 3-dimensional position $\boldsymbol{x}$, evaluate the bracket of the Cartesian components $\{p_j, A_k\}$, where $\boldsymbol{p}$ is the 3-dimensional canonical momentum vector. [7]

For a non-relativistic electron in an electromagnetic field, with a potential energy $V = e(\phi - \boldsymbol{v}\cdot\boldsymbol{A})$, derive the canonical momentum and the Hamiltonian function. [8]

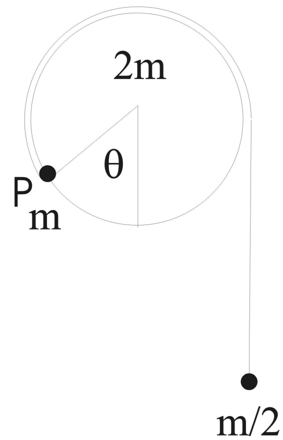
Find the value of the quantum commutator $\left[\widehat{p_j}, \widehat{A_k}\right]$, where $\widehat{\boldsymbol{p}} = -i\hbar\nabla$, and confirm the classical to quantum-mechanical analogy

$$\{f, g\} \rightarrow \frac{1}{i\hbar}\left[\widehat{f}, \widehat{g}\right]$$

where $\left[\widehat{f}, \widehat{g}\right]$ is the commutator of the corresponding operators $\widehat{f}$ and $\widehat{g}$. [5]

Derive the quantum-mechanical Hamiltonian operator and the Schrödinger equation for an electron moving in a magnetic field in the gauge $\nabla\cdot\boldsymbol{A} = 0$. [7]

2 Describe briefly how Hamilton's principle of least action leads to Lagrange's equations of motion for a dynamical system having coordinates and velocities $(q_i, \dot{q}_i)$. [4]



A uniform circular disc of mass $2m$ and radius a is mounted on an axle through its centre so that it can rotate without friction in a vertical plane. A point mass m is fixed at point P on the lower edge of the disc as shown in the figure. A light elastic string with force constant k is also attached to P, runs over the the circumference of the disc without friction, passes the topmost point and hangs down vertically. At the bottom end of the string a mass $m/2$ is attached.

Write down the Lagrangian for the system in terms of angle of rotation θ of the disc and the extension x of the string with respect to its natural length. Obtain Lagrange's equation of motion. [8]

Find the equilibrium position of the disc θ_0 and equilibrium extension x_0. [4]

Show that the natural frequencies ω of small oscillations of the system are determined by the solution to the equation

$$m^2\omega^4 - \left(\frac{5k}{2} + \frac{\sqrt{3}mg}{4a}\right) m\omega^2 + \frac{\sqrt{3}mgk}{2a} = 0$$ [8]

For the limiting case of a stiff string ($k \gg mg/a$), show that the natural frequencies approach $\omega^2 = 5k/2m$ and $\omega^2 = \sqrt{3}g/5a$, and describe the normal modes. [5]

Find appropriate expressions for the other limiting case $k \ll mg/a$ and interpret your results. [5]

3 Small spherical particles of mass m and radius a, suspended at low concentration in a liquid of viscosity η filling a vertical test tube, undergo Brownian motion. Assuming the interparticle collisions are rare and quoting the Stokes expression $6\pi\eta av$ for the friction of a dry sphere of radius a moving with velocity v, discuss the Langevin equation of motion

$$m\dot{v} = -\gamma v - mg + A(t)$$

where $\langle A(t)A(t')\rangle = \mathit{\Gamma}\delta(t - t')$. Explain the origin and effect of the three forces on the RHS. [10]

For the case $g = 0$, obtain the fluctuation-dissipation relation by calculating the equilibrium value of $\langle v^2\rangle$. [6]

Outline the derivation of the modified diffusion equation for overdamped motion of a particle in a gravitational potential

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial z}\left(\frac{mg}{\gamma} + \frac{\mathit{\Gamma}}{2\gamma^2}\frac{\partial}{\partial z}\right)P$$

where $P(z, t)$ is the particle concentration. Hence obtain the dependence of the equilibrium particle concentration on height in the test tube. [10]

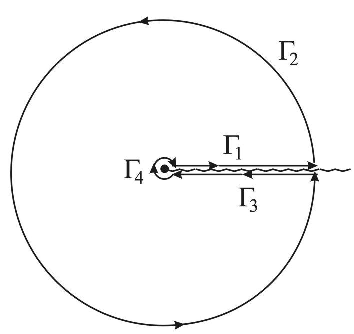
What is the vertical position of the centre of mass of the ensemble of colloid particles and how does it depend on temperature? [8]

4 Do **both** parts (a) and (b).

(a) By treating x as a complex variable and using a contour integral in which the contour is closed by a large semi-circle, or otherwise, prove that

$$\int_{-\infty}^{\infty} \frac{\sin x\ \sin(x-\alpha)}{x(x-\alpha)}\mathrm{d}x = \frac{\pi \sin\alpha}{\alpha}$$

where α is real and positive. [17]



(b) By integrating around the contour shown, or otherwise, show that, if $-1 < p < 1$ and $0 < \theta < \pi$,

$$\int_0^{\infty} \frac{x^{-p}\ \mathrm{d}x}{1 + 2x\cos\theta + x^2} = \frac{\pi}{\sin p\pi}\frac{\sin p\theta}{\sin\theta}$$ [17]

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5 Define the Fourier transform $\widetilde{f}(\omega)$ of a function $f(t)$ and write down the expression for the inverse transform. [4]

Consider an infinitely long elastic string, under tension T and having mass per unit length ρ. There is a one-dimensional transverse displacement field $u(x,t)$. Derive the Lagrangian density of the system and its equations of motion. [8]

Assume that damping cannot be neglected and is proportional to velocity $\dot{u}$. Derive the new equation of motion for the string, either by directly amending the classical wave equation, or via the Rayleigh dissipation function. [6]

Explain how Fourier transforms can be used to analyse the motion of the string. [6]

Hence or otherwise find an expression for the real time/space form of the Green's function and explain the contour integration procedure and its implications for causality. [10]

6 Describe the concept of transition probability of classical discrete and continuous processes. [10]

Examine the role of transition probability in establishing equilibrium and determining kinetic relaxation. [14]

Via the path-integral formalism, or otherwise, show that the most probable trajectory for system evolution corresponds to the path that extremises the classical action, using Brownian motion as a simple practical application. [10]