Bose-Hubbard Models with Gauge Fields: Frustrated Quantum Spins

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[Gunnar Möller & NRC, PRL **103**, 105303 (2009) & unpublished]

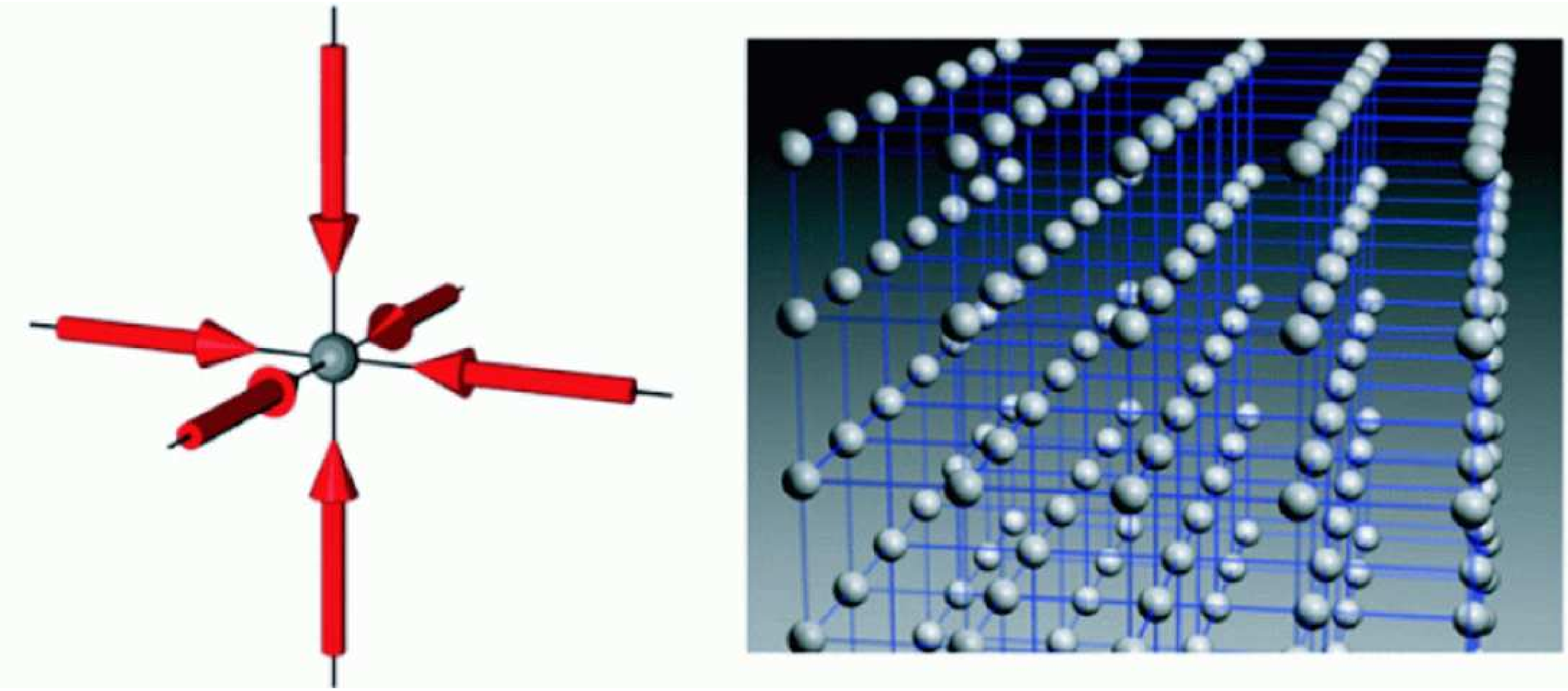
Overview

- Strongly Correlated Phases of Atomic Bose Gases
- Bose-Hubbard Model + Gauge Fields
- 1) Uniform Flux: Strongly Correlated Phases
- 2) Staggered Flux: Condensed Phases
- Summary

Strongly Correlated Phases of Atomic Bose Gases

(1) Optical Lattice

[Bloch, Dalibard & Zwerger, RMP **80**, 885 (2008)]



Bose-Hubbard model

[Jaksch *et al.*, PRL **81**, 3108 (1998)]

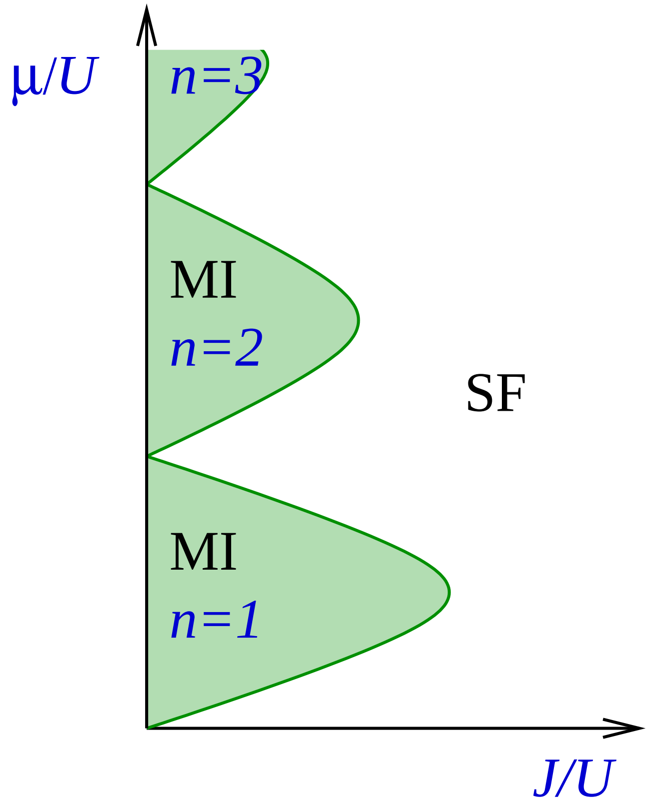
$$\hat{H} = -J \sum_{\langle i,j \rangle} \left[\hat{a}_i^\dagger \hat{a}_j + h.c. \right] + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

Strongly correlated regime for $U/J \gg 1$
at particle density $n \sim 1$.

$T = 0$: competition between

- superfluid (BEC)
- Mott insulators, at $n = 1, 2, \ldots$

[Fisher *et al.*, PRB **40**, 546 (1989)]

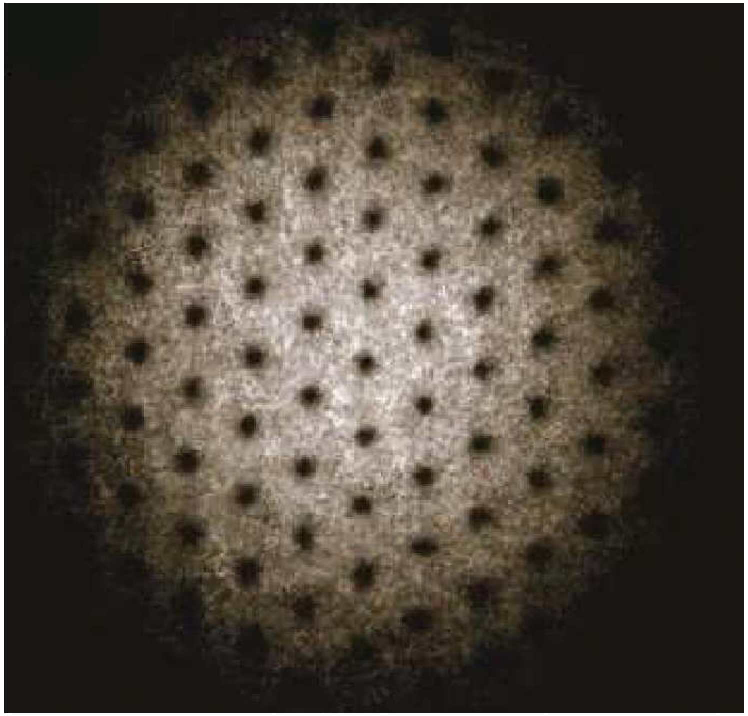


Transition to Mott insulator observed in experiment [Greiner *et al.*, Nature **415**, 39 (2002); . . .]

Strongly Correlated Phases of Atomic Bose Gases

(2) Rapid Rotation

[Bloch, Dalibard & Zwerger, RMP **80**, 885 (2008)]



Rotation frequency, Ω

Quantized vortices

Vortex density $n_{\rm v} = \frac{2M\Omega}{h}$

[Coddington *et al.* [JILA], PRA **70**, 063607 (2004)]

Strong correlation regime for $\Omega \to \omega_\perp$

Filling Factor $\nu \equiv \frac{n_{\mathrm{2d}}}{n_{\mathrm{v}}}$

Critical filling factor $\nu_{\mathrm{c}} \simeq 6$

- $\nu > \nu_{\mathrm{c}}$: Vortex Lattice (BEC)

- $\nu < \nu_{\mathrm{c}}$: *Bosonic* versions of fractional quantum Hall states:
Laughlin, hierarchy/CF, Moore-Read & Read-Rezayi phases, smectic +...?

[For a review, see: NRC, Adv. Phys. **57**, 539 (2008)]

Experimental challenges:

- the interaction scale at $\nu \sim 1$ is small;

- rotating gas susceptible to "heating" by static perturbations.

Bose-Hubbard Model + Gauge Fields

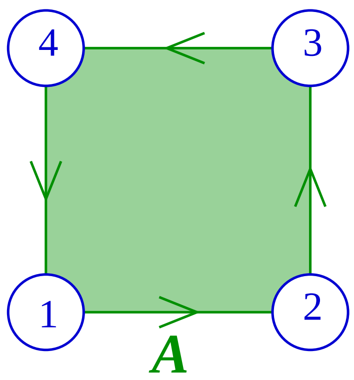
[Jaksch & Zoller (2003); Mueller (2004); Sørensen, Demler & Lukin (2005); Gerbier & Dalibard (2010)]

Bose-Hubbard model with "magnetic field" (2D square lattice)

$$\hat{H} = -J \sum_{\langle i,j \rangle} \left[\hat{a}_i^\dagger \hat{a}_j e^{iA_{ij}} + h.c. \right] + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

Gauge invariant fluxes $\quad n_{\mathrm{v}} = n_\phi = \frac{1}{2\pi} [A_{12} + A_{23} + A_{34} + A_{41}]$

$(0 \leq n_{\mathrm{v}} < 1)$



Vortex/flux density, $n_{\mathrm{v}} \sim 1$

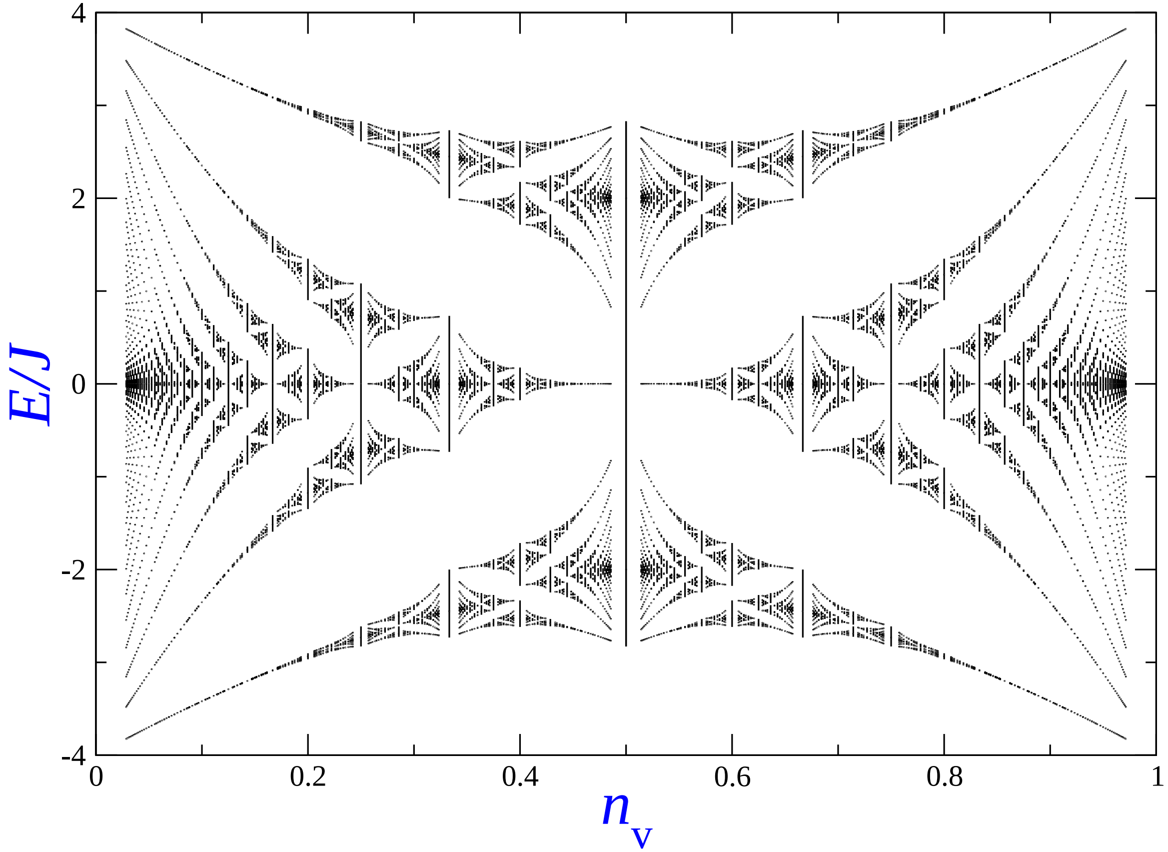
Particle density, $n \sim 1$

Interaction strength, $U/J \gtrsim 1$

What are the groundstates for interacting bosons?

Uniform n_{v}: Single particle spectrum is the "Hofstadter butterfly"

[Harper, Proc. Phys. Soc. Lond. A **68**, 874 (1955); Hofstadter, PRB **14**, 2239 (1976)]



$n, n_{\mathrm{v}} \ll 1 \Rightarrow$ continuum limit [Sørensen, Demler & Lukin, PRL (2005); Hafezi *et al.*, PRA (2007)]

Are there new strongly correlated phases on the lattice for $n \sim n_{\mathrm{v}} \sim 1$?

Hard-core limit, $U \gg J$

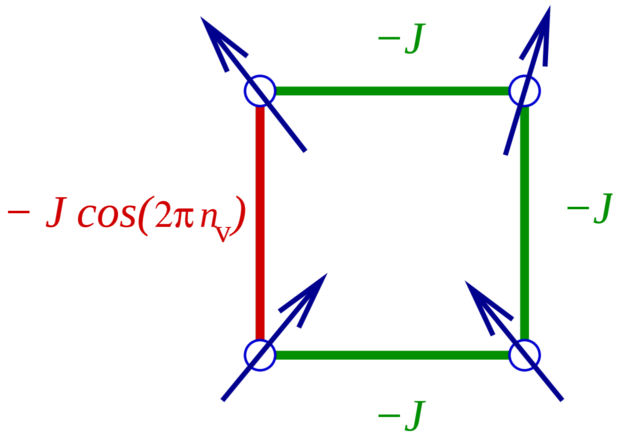
$0 \leq n_i \leq 1$ $\Rightarrow$spin-1/2 system: $\hat{s}_i^z = \hat{n}_i - \frac{1}{2}$, $\hat{s}_i^+ = \hat{a}_i^\dagger$, $\hat{s}_i^- = \hat{a}_i$

$$\hat{H} = -J \sum_{\langle i,j \rangle} \left[\hat{s}_i^+ \hat{s}_j^- e^{iA_{ij}} + h.c. \right] - \mu \sum_i \hat{s}_i^z + \text{const.}$$

Frustrated quantum magnet.

Mean-field theory: $\vec{s} = S(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$H = -2JS^2 \sum_{\langle i,j \rangle} \sin\theta_i \sin\theta_j \cos(\phi_i - \phi_j + A_{ij}) - \mu S \sum_i \cos\theta_i$



Are there "spin-liquid" phases? [NB Energy scale, J, is large!]

(1) Uniform Flux: Strongly Correlated Phases

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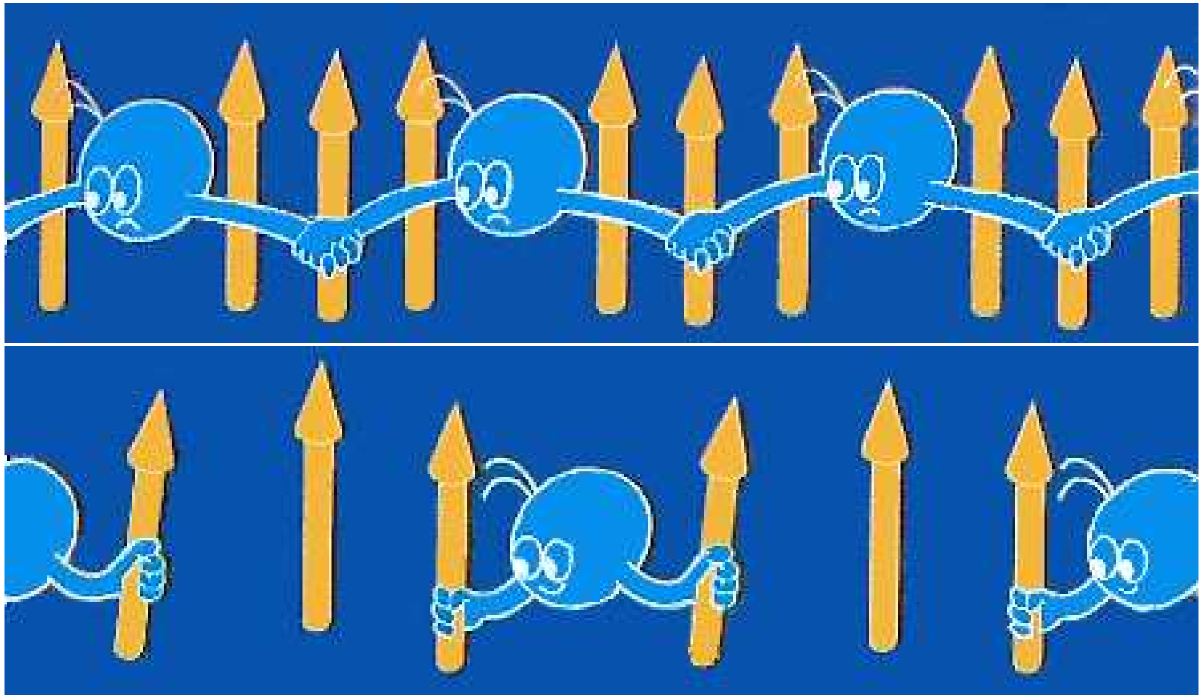
Composite Fermions [Jain, Read, Girvin, Fradkin,. . .]

Interacting electrons in magnetic field $\Rightarrow$non-interacting *composite fermions*.



[Illustration by Kwon Park]

Composite fermion = bound state of an electron with two flux quanta.

Composite Fermions

Rapidly rotating bosons in the continuum

Composite fermion = a bound state of a boson with *one vortex*.

[NRC & Wilkin, PRB **80**, 16279 (1999)]

$$\Psi_{\mathrm{B}}(\{\boldsymbol{r}_i\}) \propto \mathcal{P}_{LLL} \prod_{i<j} (z_i - z_j)\, \psi_{\mathrm{CF}}(\{\boldsymbol{r}_i\}) \qquad \left[z = (x+iy)/\ell \quad \ell = 1/\sqrt{2\pi n_{\mathrm{v}}}\right]$$

$$n_{\mathrm{v}}^{\mathrm{CF}} = n_{\mathrm{v}} - n$$

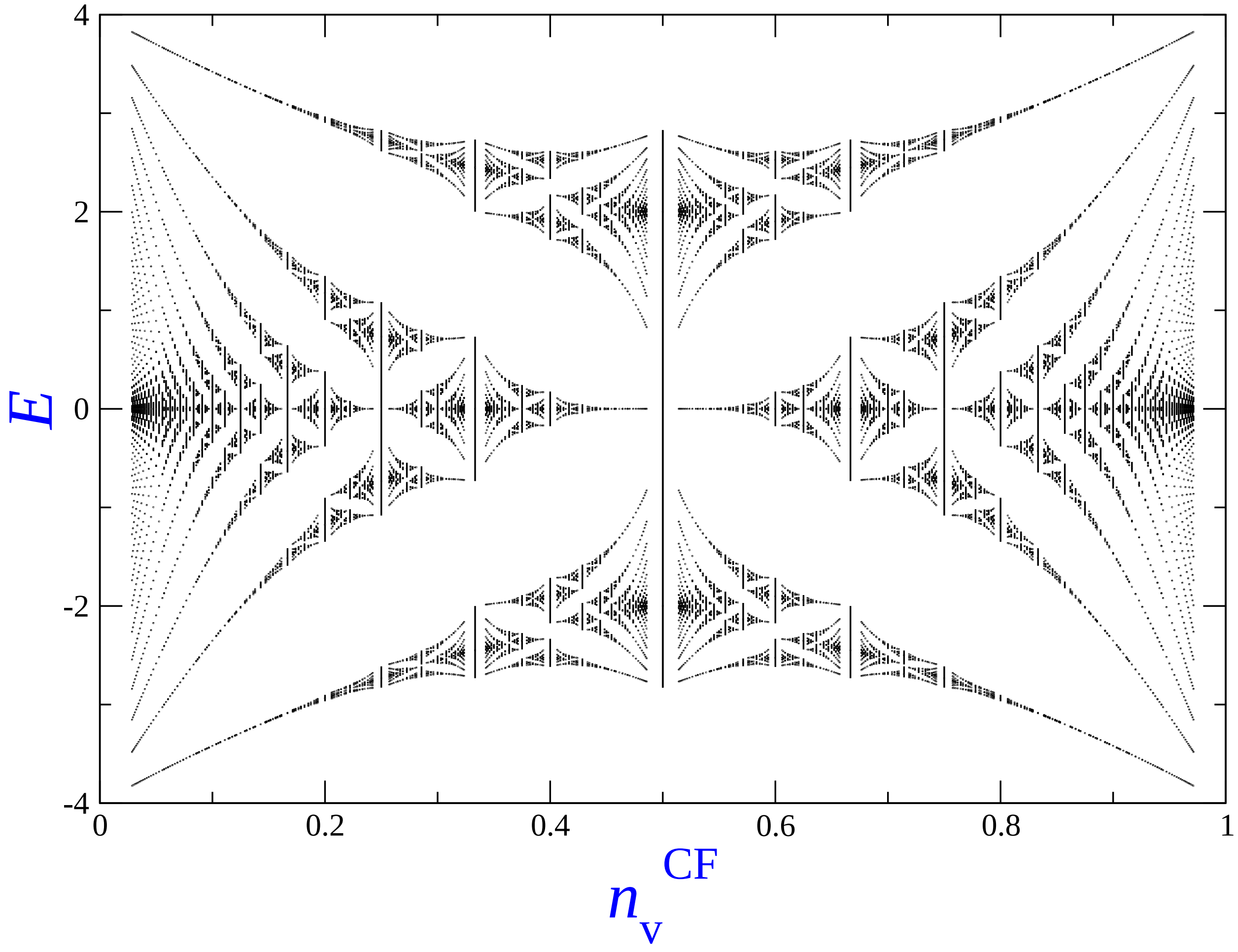
CFs fill p Landau levels for

$$\frac{n}{n_{\mathrm{v}}^{\mathrm{CF}}} = \pm p \qquad \Rightarrow \qquad \nu = \frac{n}{n_{\mathrm{v}}} = \frac{p}{p \pm 1}$$

$\Rightarrow$(trial) incompressible states of interacting bosons, describe exact groundstates well for $\nu = 1/2, 2/3, (3/4)$

[NRC & Wilkin, PRB (1999); Regnault & Jolicoeur, PRL (2003); . . .]

Lattice: CF spectrum is the "Hofstadter butterfly"

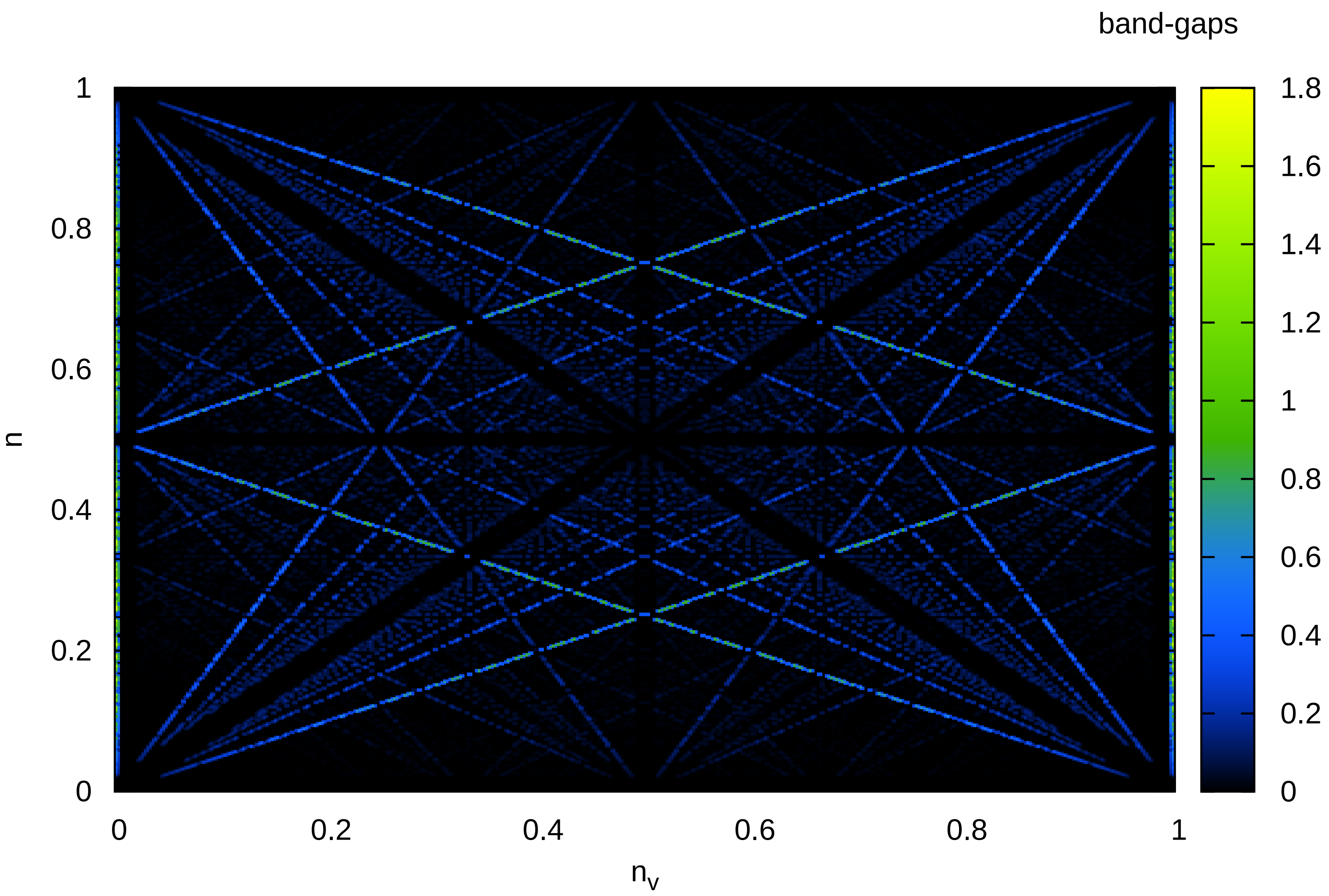


Filled band of CFs at (n, n_v^{CF}) $\Rightarrow$trial incompressible state of bosons at (n, n_v)

There can exist incompressible states with no counterpart in the continuum

Gaps for non-interacting CFs

[G. Möller & NRC, PRL **103**, 105303 (2009)]



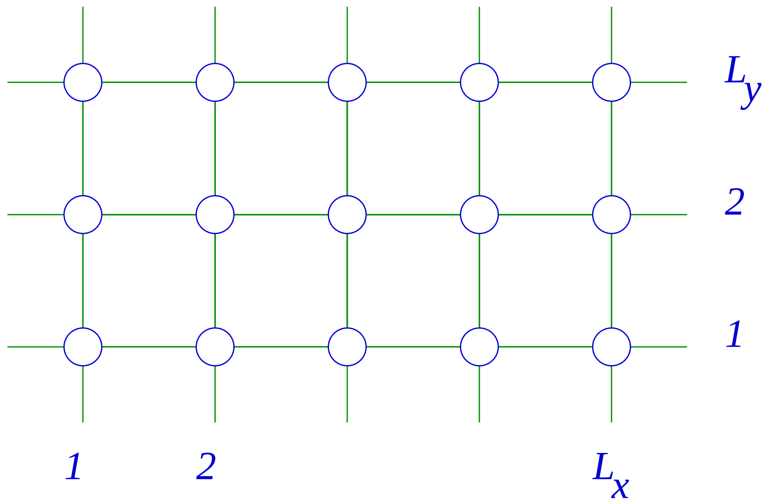
Do these new phases describe the exact groundstates?

Numerical Methods

- Exact Diagonalization

$L_x \times L_y$ square lattice, with periodic boundary conditions (torus).

$$N = nL_xL_y$$
$$N_\mathrm{v} = n_\mathrm{v}L_xL_y$$



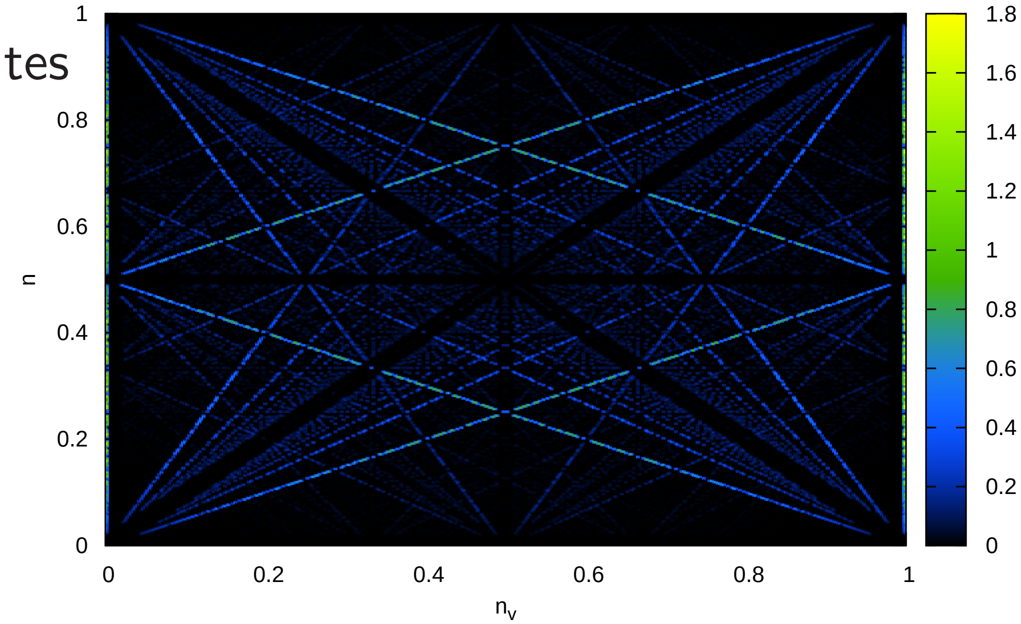
- Low-energy spectrum (Lanczos) for hard-core interactions.

- Limited by finite size effects, $N \leq 6$.

CF states stabilized by the lattice

[G. Möller & NRC, PRL **103**, 105303 (2009)]

Evidence for strongly correlated states at a series of these new cases.



On $n_\mathrm{v} = \frac{1}{2}(1-n)$: Groundstate is consistent with the CF state for $n \lesssim 1/5$.

- Uncondensed.
- Large overlaps with a trial CF state.
- Correct groundstate degeneracy on the torus (1).
- Correct Chern number (2).

Evidence for wider applicability of CF ansatz.

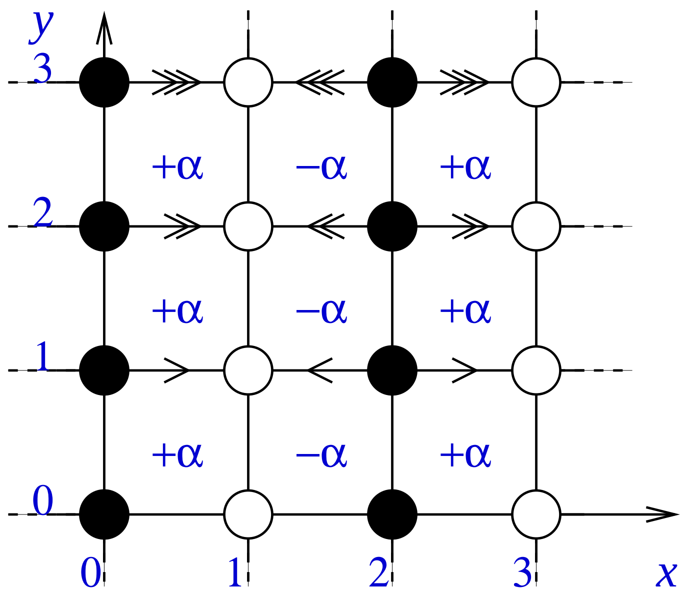
(2) Staggered Flux: Condensed Phases

(2) Staggered Flux: Condensed Phases

Staggered Fluxes

Easiest to generate in scheme of [Gerbier & Dalibard (2010)]

$$A_{ij} = 2\pi\alpha y_i(x_i - x_j) \times (-1)^{x_i}$$



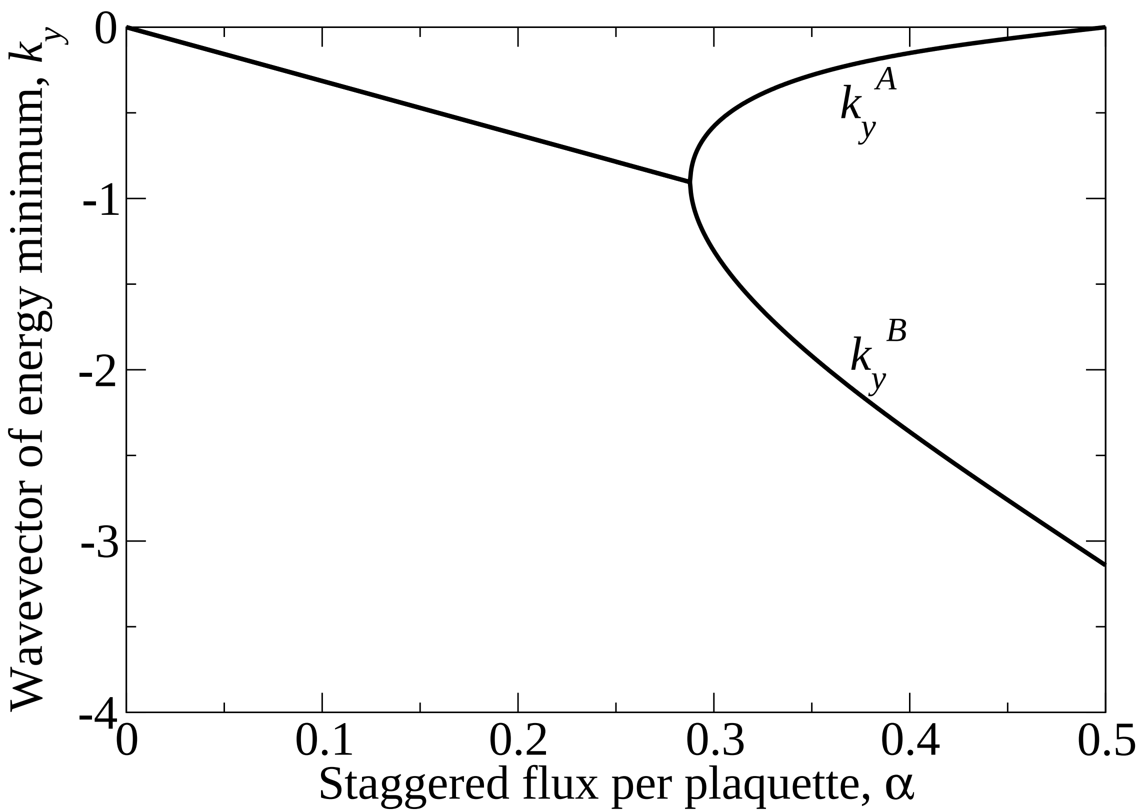
cf. checkerboard staggered fluxes [Lim, Morais Smith & Hemmerich, PRL **100**, 130402 (2008)]

Special case for $\alpha = 1/2$: gauge-equivalent to uniform $n_{\mathrm{v}} = 1/2$
time-reversal symmetry, $e^{iA_{ij}} = \pm 1$
"fully frustrated" magnet [Villain, J Phys C **10**, 1717 (1977)]

Staggered Fluxes: Single Particle Spectrum

$$\psi_i = e^{i(k_x x_i + k_y y_i)} \times \begin{cases} \psi_e & x_i \text{ even} \\ \psi_o e^{i2\pi\alpha y_i} & x_i \text{ odd} \end{cases}$$

Groundstate is two-fold degenerate for $\alpha > \alpha_c \simeq 0.29$.



Gross-Pitaevskii Mean-Field Theory

Fully condensed state

$$|\Psi\rangle = \left(\sum_i \psi_i^c \, \hat{a}_i^\dagger \right)^N |0\rangle$$

Energy per particle ($N \gg 1$)

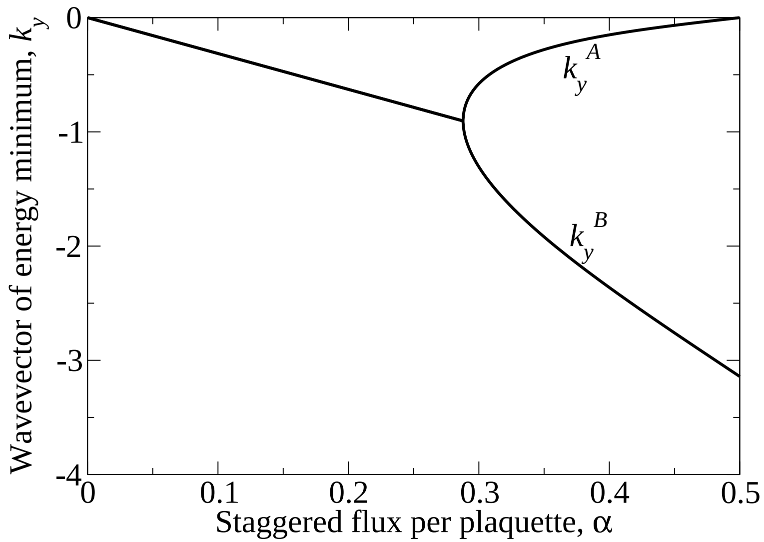
$$\frac{\langle \hat{H} \rangle}{N} \;\; = \;\; -J \sum_{\langle i,j \rangle} \left[\psi_i^{c*} \psi_j^c e^{iA_{ij}} + \psi_j^{c*} \psi_i^c e^{iA_{ji}} \right] + \frac{U}{2} N \sum_i |\psi_i^c|^4$$

<u>Weak Coupling, $nU \ll J$</u>

$\psi^c \simeq A\psi_{k_A} + B\psi_{k_B}$ $\quad\Rightarrow\quad$ minimize $\sum_i |\psi_i^c|^4$

$|A|, |B| \neq 0 \Rightarrow$ broken translational invariance

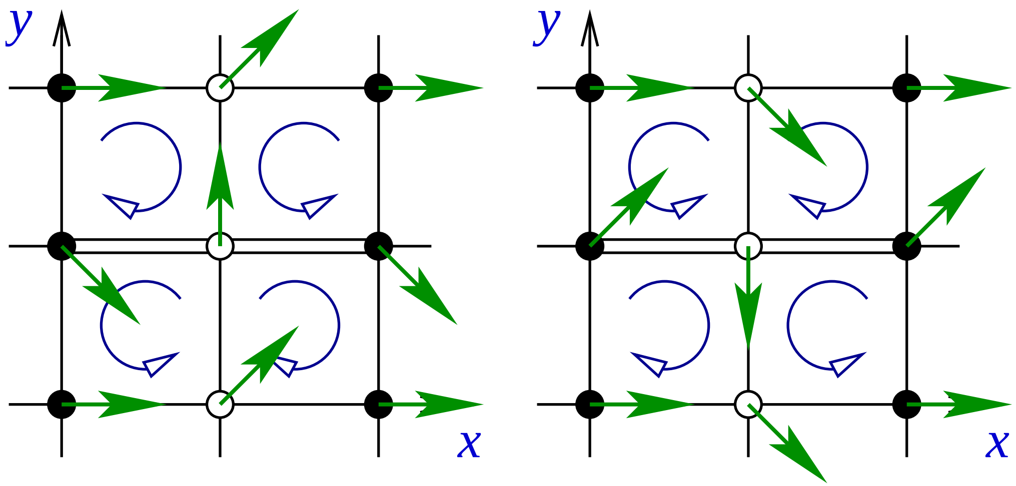
Mean-Field Theory: Results



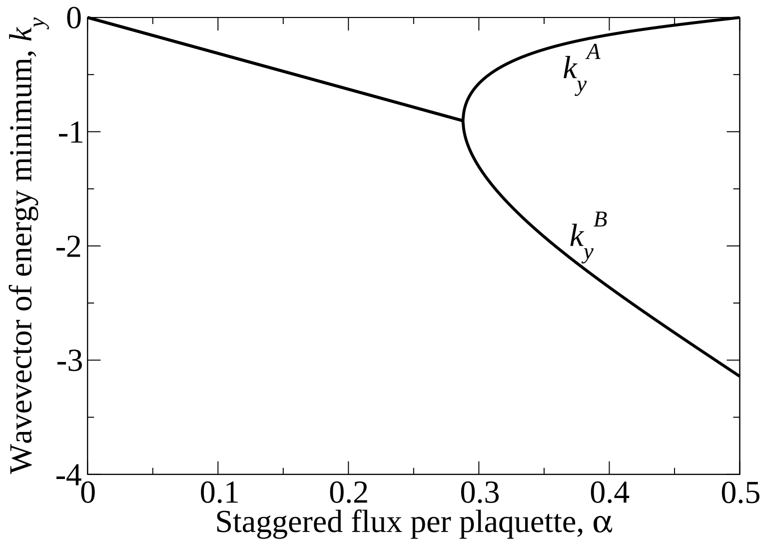
$\underline{\alpha = 1/2}$ $\qquad k_y^A - k_y^B = \pi$

$$\psi^c_\pm = \frac{1}{\sqrt{2}}\left[\psi_{k_A} \pm i\psi_{k_B}\right]$$

- Broken translational invariance ($\Delta y = 2$)
- Broken time-reversal symmetry
- "Staggered Flux" Phase



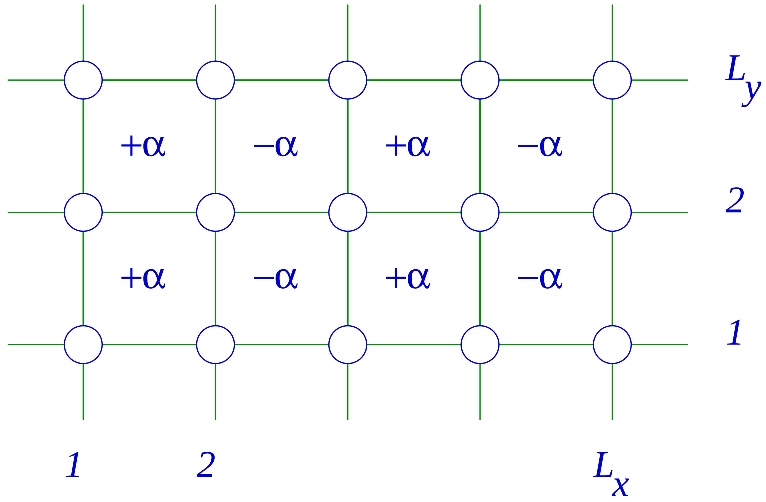
$\underline{\alpha = 0.389}$ $\quad k_y^A - k_y^B = 2\pi/3$



$$\psi^c = \frac{1}{\sqrt{2}}\left[\psi_{k_A} + e^{i\phi}\psi_{k_B}\right]$$

- Interaction energy independent of $\phi \Rightarrow$ "Goldstone" mode.
- Increasing nU/J selects $\phi = 0, \pm 2\pi/3$.
- Broken translational invariance ($\Delta y = 3$)

Numerical Methods



- Exact Diagonalization

$L_x \times L_y$ square lattice, with periodic boundary conditions (torus).

$$N = nL_xL_y$$

- Low-energy spectrum (Lanczos).

- Limited by finite size effects, $N \leq 12$.

Condensed Fraction

Single-particle density matrix of the groundstate

$$\rho_{ij} = \langle \Psi_0 | \hat{a}_i^\dagger \hat{a}_j | \Psi_0 \rangle$$

"Simple" BEC: One eigenvalue, λ_0, is of order N.

Condensate fraction

[Yang, Rev. Mod. Phys. **34**, 694 (1962)]

$$x_c \equiv \frac{\lambda_0}{N}$$

Condensate wavefunction from eigenvector.

Condensed Fraction with Symmetry Breaking

If the condensed state has D-fold degeneracy, related by symmetries of the underlying Hamiltonian (e.g. $D = 2$ for staggered flux phase)

- Exact groundstate is "fragmented", with D large eigenvalues.
- Lowest energy spectrum breaks into D-fold (quasi-) degenerate states.

[e.g. Mueller, Ho, Ueda & Baym, PRA (2006)]

Condensed Fraction with Symmetry Breaking

General Numerical Method

(1) From the energy spectrum, identify an emerging quasi-degenerate set of groundstates $|\Psi_0^\mu\rangle$ ($\mu = 1, D$).

(2) Construct $|\Psi^c\rangle \equiv \sum_{\mu=1}^{D} c_\mu |\Psi_0^\mu\rangle$.

(3) Maximize the eigenvalue of the resulting single-particle density matrix

$$X_c \equiv \max_{c_\mu} \left[x_c(c_\mu) \right]$$

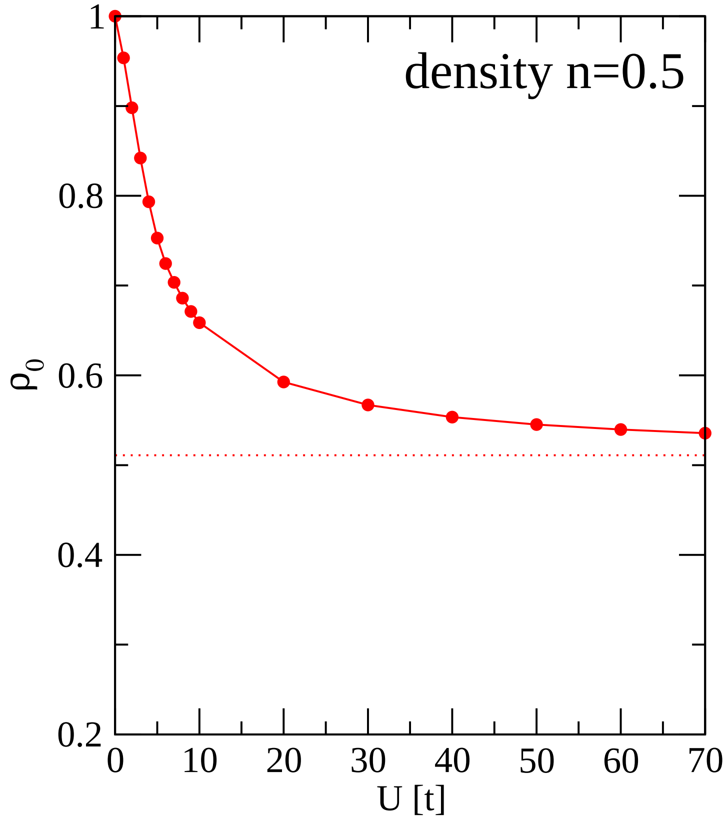
$\Rightarrow$condensate fraction and condensate wavefunction.

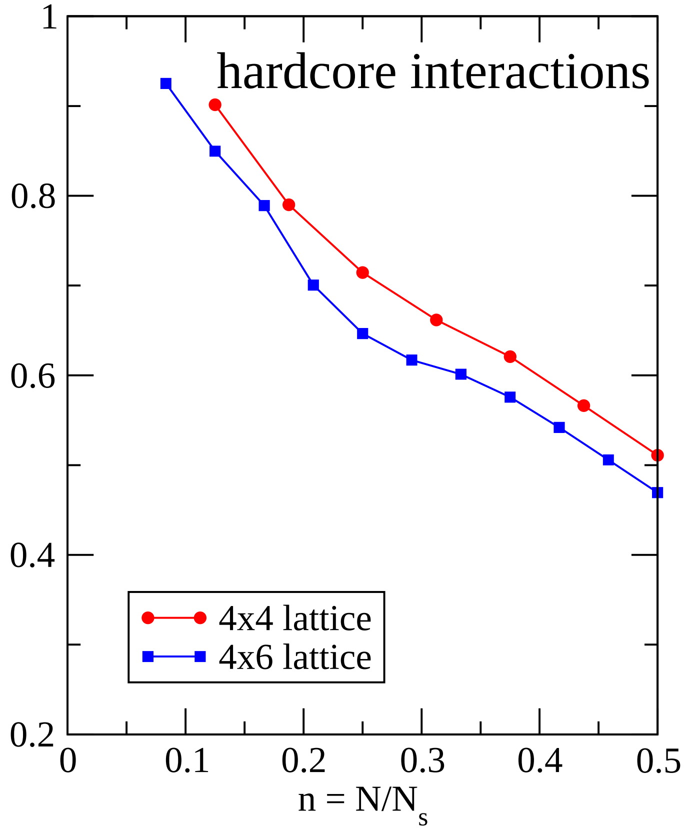
Condensate fraction for $\alpha = 1/2$

MFT   Staggered flux state ($D = 2$), $\psi^c_\pm$.

Exact results

- The inferred condensed state is *exactly* the mean-field state.
- Condensate depletion up to $\sim 50\%$.





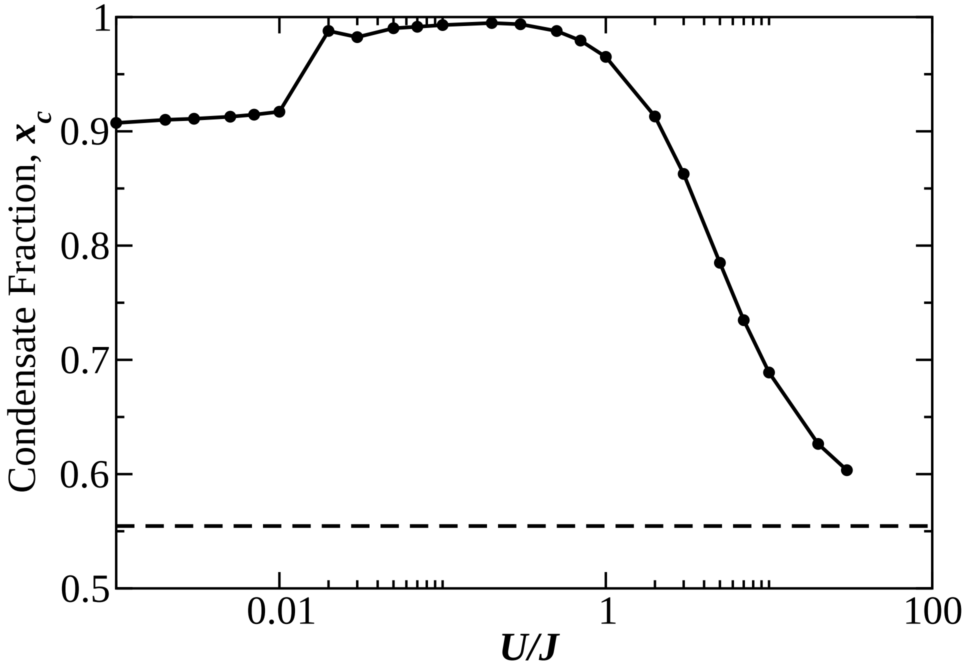
Condensate fraction for $\alpha = 0.389$

MFT

- $nU \ll J$ continuous degeneracy (additional Goldstone mode)
- Increasing nU/J selects $\phi = 0, \pm 2\pi/3$. ($D = 3$)

Exact results assuming $D = 3$

- Qualitative agreement with MFT state for $nU/J \gtrsim 0.01$



$L_x \times L_y = 3 \times 4$
$N = 6$
$n = 1/2$

Expansion Imaging

Expansion under $\hat{H}_{\rm free}$: no potentials, no gauge field, (neglect interactions)

After time t

$$n(\boldsymbol{x}) = (M/\hbar t)^3 \, |\tilde{w}(\boldsymbol{k})|^2 G(\boldsymbol{k})$$

$\boldsymbol{k} = M\boldsymbol{x}/\hbar t$, $\tilde{w}(\boldsymbol{k})$ from Wannier orbital [Bloch, Dalibard & Zwerger, RMP (2008)]

$$\begin{aligned} G(\boldsymbol{k}) &= \sum_{i,j} e^{i\boldsymbol{k}\cdot\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)} \langle \hat{a}_i^\dagger \hat{a}_j \rangle \\ &\simeq \sum_{i,j} e^{i\boldsymbol{k}\cdot\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)} \psi_i^{c*} \psi_j^c \qquad \text{[pure condensate]} \end{aligned}$$

1. Gauge Invariance

$$\hat{H} = -J \sum_{\langle i,j \rangle} \left[\hat{a}_i^\dagger \hat{a}_j e^{iA_{ij}} + h.c. \right] + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

Change of gauge $A_{ij} \rightarrow A'_{ij} = A_{ij} + S_i - S_j$

Removed by the redefinition $\quad \hat{a}_i \rightarrow \hat{a}'_i \equiv \hat{a}_i e^{-iS_i}$

All physical properties of $\hat{H}$ (spectrum, response functions, ...) are gauge-invariant.

Density matrix

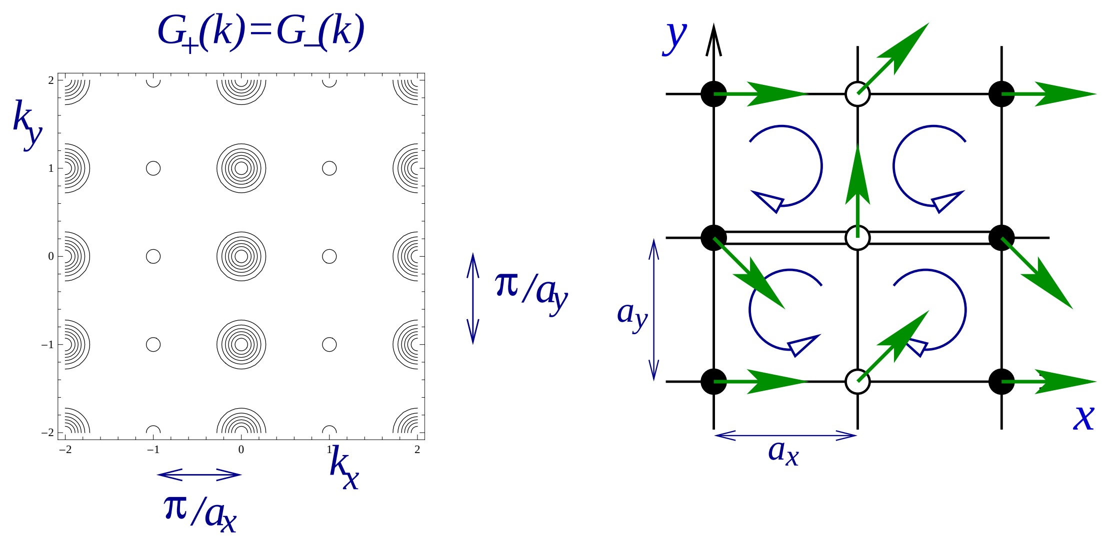
$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle \rightarrow \langle \hat{a}_i'^\dagger \hat{a}'_j \rangle = e^{i(S_i - S_j)} \langle \hat{a}_i^\dagger \hat{a}_j \rangle$$

The expansion image, under $\hat{H}_{\text{free}}$, is sensitive to S_i $\Rightarrow$ “gauge-dependent”.

Example: Staggered flux phase

- In our gauge, the two states $\psi^c_\pm$ are related by $\psi^c_- = (\psi^c_+)^*$

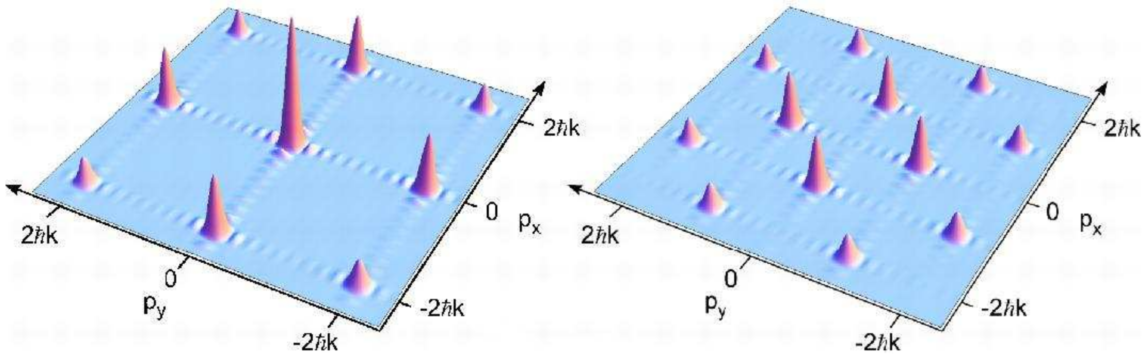
$\Rightarrow G_+(\boldsymbol{k}) = G_-(\boldsymbol{k})$



- In the gauge of [Lim, Morais Smith & Hemmerich, PRL **100**, 130402 (2008)]

$\psi^c_+ = 1$

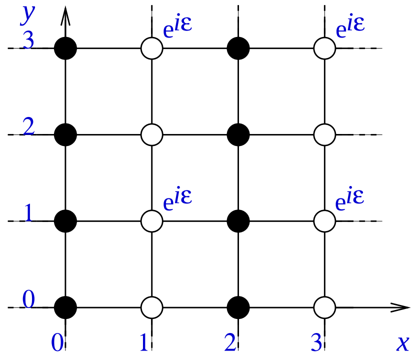
$\psi^c_- = (-1)^x e^{i\pi/2[\mathrm{mod}(x+y,2)]}$



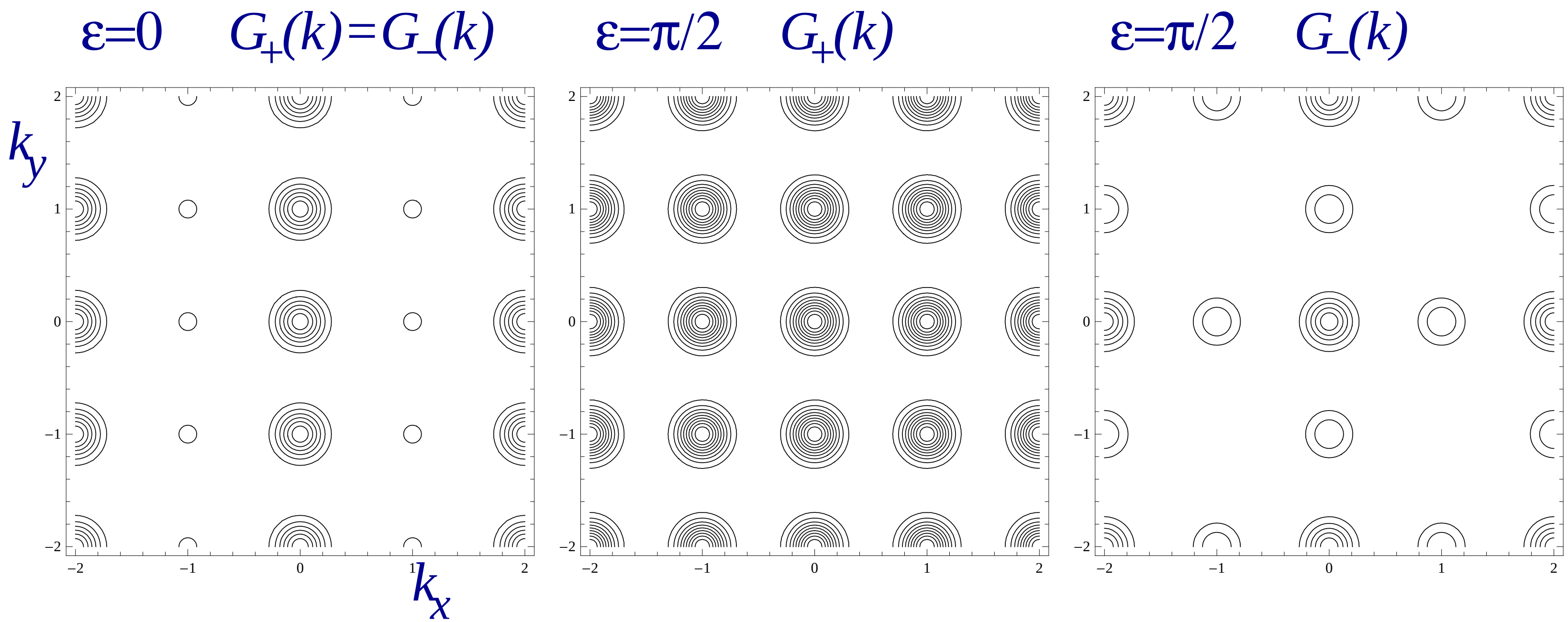
2. Phase Imprinting

Use potentials/optical dressing to imprint of phases e^{iS_i} prior to release.

e.g. $S_i = \epsilon \times \mathrm{mod}(x_i, 2)\mathrm{mod}(y_i, 2)$



$\epsilon \neq 0$ distinguishes staggered flux phases, $G_+(\boldsymbol{k}) \neq G_-(\boldsymbol{k})$



Summary

• Atomic Bose gases on a lattice with gauge fields offer the possibility to explore interesting strong correlation phenomena:
– the FQHE of bosons (at large energy scales);
– the interplay of the FQHE and lattice periodicity;
– equivalent to a class of frustrated quantum magnets (energy scale J).

<u>(1) Uniform Flux</u>

• A generalized composite fermion construction leads to the prediction of strongly correlated phases of bosons, including states which are stabilized by the lattice.

• We find numerical evidence for the appearance of these phases for several of the predicted cases. This shows a wider applicability of the CF construction.

<u>(2) Staggered Flux</u>

• Allowing for translational symmetry breaking, we find that exact diagonalization studies are consistent with the mean-field groundstates.

• Expansion images are gauge dependent. Imprinting phase patterns before expansion can give useful additional information.