Weak Measurements: Wigner-Moyal and Bohm in a New Light.

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Weak Measurements.


2. What do we measure? Weak values defined by

\[ A_W(\psi,\phi) = \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} \in \mathbb{C} \]

3. Will show that \( P_W(\psi,x) \) is related to the Bohm momentum

\[ P_W^2(\psi,x) \] is related to the Bohm energy and quantum potential. [Leavens, Found. Phys., 35 (2005) 469-91]


5. Will indicate how to combine the Clifford with the Moyal algebra.

6. Discuss what may lie behind the formalism-- Process.

Acknowledgements

Ernst Binz, Maurice de Gosson, Bob Callaghan and David Robson.
Weak measurements.

Why the interest?

Photon ‘trajectories’.

Schrödinger particle ‘trajectories’

Experimental--Photons.

Theory--Schrödinger particle.

\[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0 \]

\[ E_B = -\frac{\partial S}{\partial t} \quad P_B = \nabla S \]

[Kocsis, Braverman, Ravets, Stevens, Mirin, Shalm, Steinberg, Science 332, 1170 (2011)]

[Prosser, IJTP, 15, (1976) 169]

[Philippidis, Dewdney and Hiley, Nuovo Cimento 52B, 15-28 (1979)]
Weak Measurement: the principle.

\[ H_I = \lambda(t) \hat{A}_s \hat{B}_d \quad |\text{initial}\rangle = |\psi_s\rangle |\Psi_d\rangle \quad \text{and} \quad |\text{final}\rangle = e^{i \int \lambda(t) \hat{A}_s \hat{B}_d dt} |\psi_s\rangle |\Psi_d\rangle \]

\[ \int \lambda(t) dt = 1 \quad |\text{final}\rangle = e^{i \hat{A}_s b} |\psi_s\rangle |\Psi_d\rangle \]

Form the transition probability amplitude

\[ T_{\phi-\psi} = \langle b_d | \langle \phi_s | |\text{final}\rangle \rangle \quad \langle b_d | \hat{A} | \Psi_d \rangle = \hat{A} \Psi_d (b) \]

\[ T_{\phi-\psi} = \langle \phi_s | e^{i \hat{A} b} | \psi_s\rangle \Psi_d (b) = \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} \langle \phi_s | \hat{A}^n | \psi_s\rangle \Psi_d (b) = \langle \phi_s | \psi_s\rangle \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} \frac{\langle \phi_s | \hat{A}^n | \psi_s\rangle}{\langle \phi_s | \psi_s\rangle} \Psi_d (b) \]

\[ T_{\phi-\psi} = \langle \phi_s | \psi_s\rangle \left[ e^{ib \langle A \rangle_W} + \sum_{n=2}^{\infty} \frac{(ib)^n}{n!} [\langle A^n \rangle_W - \langle A \rangle_W^n] \right] \Psi_d (b) \]

small

Post select with \( b = x \)

\[ T_{\phi-\psi} = \langle \phi_s | \psi_s\rangle e^{ix \langle A \rangle_W} \Psi_d (x) \]

Choose \( \Psi_d (x) = \exp \left[ -\frac{x^2}{4(\Delta x)^2} \right] \) and take the imaginary part of \( \langle A \rangle_W \) we find

\[ T_{\phi-\psi} \propto e^{-x \langle A \rangle_{IW}} \exp \left[ -\frac{x^2}{4(\Delta x)^2} \right] \times \exp \frac{-[x + 2(\Delta x)^2 \langle A \rangle_{IW}]^2}{4(\Delta x)^2} \]

Centre of Gaussian in x-space shifted by amount \( \propto \langle A \rangle_{IW} \)

Centre of Gaussian in p-space shifted by amount \( \propto \langle A \rangle_{RW} \)

Weak Experiment cont.

von Neumann measurement.  
Single measurement ---- collapse of the wave function.  
\[ \sum_j \Psi_j \rightarrow \Psi_r \]

Weak measurement.

Statistical measurement producing a phase shift in the distribution of final results.

Actual experimental arrangement for photon trajectories.

\[ |\Psi\rangle = |D\rangle_{\text{polarizer}} |\psi\rangle_{\text{path}} \]

\[ H = g\hat{P}_x \hat{S}_1 \]

\[ \hat{S}_1 = (|H\rangle\langle H| - |V\rangle\langle V|)/2 \]

[Kocsis, Braverman, Ravets, Stevens, Mirin, Shalm, Steinberg,  
Weak values. \( \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} \)

How do they appear in the formalism?

\[
\langle \psi | A | \psi \rangle = \sum \langle \psi | \phi_j \rangle \langle \phi_j | A | \psi \rangle
\]

where \( | \phi_j \rangle \) form a complete orthonormal set.

Then

\[
\langle \psi | A | \psi \rangle = \sum \langle \psi | \phi_j \rangle \left( \frac{\langle \phi_j | \psi \rangle}{\langle \phi_j | \psi \rangle} \right) \langle \phi_j | A | \psi \rangle = \sum \rho_j \frac{\langle \phi_j | A | \psi \rangle}{\langle \phi_j | \psi \rangle}
\]

Special case:

If \( A | \phi_j \rangle = a_j | \phi_j \rangle \)

\[
\langle \psi | A | \psi \rangle = \sum \rho_j a_j
\]

Well known result!

Eigenvalue!

Remember \( \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} \) is a complex number.

But what does it mean in general?

It is clearly a transition probability amplitude.
Weak values when $\hat{P}$ is involved.

Consider

$$\langle x | \hat{P} | \psi(t) \rangle = \int \langle x | \hat{P} | x' \rangle \langle x' | \psi(t) \rangle dx' = -i \nabla_x \psi(x, t)$$

Write $\psi(x, t) = R(x, t)e^{iS(x, t)}$ then

$$\frac{\langle x | \hat{P} | \psi(t) \rangle}{\langle x | \psi(t) \rangle} = \nabla_x S(x, t) - i \nabla_x \rho(x, t) / 2 \rho(x, t)$$

with $\rho(x, t) = |\psi(x, t)|^2$

Bohm momentum.

osmotic momentum.

Bohm velocity:

$$v_B(x, t) = \frac{\nabla_x S(x, t)}{m} = \frac{1}{2m} \left[ \frac{\langle \psi(t) | \hat{P} | x \rangle}{\langle \psi(t) | x \rangle} + \frac{\langle x | \hat{P} | \psi(t) \rangle}{\langle x | \psi(t) \rangle} \right]$$

Osmotic velocity:

$$v_O(x, t) = \frac{1}{2m} \frac{\nabla_x \rho(x, t)}{\rho(x, t)} = \frac{1}{2m} \left[ \frac{\langle \psi(t) | \hat{P} | x \rangle}{\langle \psi(t) | x \rangle} - \frac{\langle x | \hat{P} | \psi(t) \rangle}{\langle x | \psi(t) \rangle} \right]$$

In terms of one expression

$$[[P_W]]_{\pm} = \left[ \frac{\langle \psi(t) | \hat{P} | x \rangle}{\langle \psi(t) | x \rangle} \pm \frac{\langle x | \hat{P} | \psi(t) \rangle}{\langle x | \psi(t) \rangle} \right] = \frac{\psi^*(x, t) \hat{P}_x \psi(x, t) \pm \psi(x, t) \hat{P}_x \psi(x, t)}{\rho(x, t)}.$$ 

More simply for Schrödinger

$$[[P_W]]_{\pm} = \frac{i}{\rho(x, t)}[\nabla_x \psi^*(x, t)] \psi(x, t) \mp \psi^*(x, t)(\nabla_x \psi(x, t))$$

then

$$-i\rho[[P_x W]]_+ = [\nabla_x \psi^*(x)] \psi(x) - \psi^*(x)[\nabla_x \psi(x)] = \psi^*(x) \nabla_x \psi(x)$$

$$-i\rho[[P_x W]]_- = [\nabla_x \psi^*(x)] \psi(x) + \psi^*(x)[\nabla_x \psi(x)] = \nabla_x (\psi^*(x) \psi(x)) = \nabla_x \rho(x).$$
Remark 1: Relation to Nelson.

Mean forward derivative: \[ Dx(t) = \lim_{\Delta t \to 0^+} \frac{E_i \frac{x(t + \Delta t) - x(t)}{\Delta t}}{E_i(x)} \]

Mean backward derivative: \[ D_\ast x(t) = \lim_{\Delta t \to 0^+} \frac{E_i \frac{x(t) - x(t - \Delta t)}{\Delta t}}{E_i(x)} \]

With these construct a forward velocity \( b(x, t) \) and backward velocity \( b_\ast(x, t) \):

\[ \frac{b(x, t) + b_\ast(x, t)}{2} = v_B(x, t) = \frac{\nabla_x S(x, t)}{m} \]

\[ b(x, t) - b_\ast(x, t) = v_O(x, t) = \frac{1}{2m} \frac{\nabla_x \rho(x, t)}{\rho(x, t)} \]
Remark 2: Relation to Energy-Momentum Tensor.

\[ T^{\mu\nu} = -\left\{ \frac{\partial L}{\partial (\partial^\mu \psi)} \partial^\nu \psi + \frac{\partial L}{\partial (\partial^\mu \psi^*)} \partial^\nu \psi^* \right\} \]

Take the Schrödinger Lagrangian:

\[ L = -\frac{1}{2m} \nabla \psi^* \cdot \nabla \psi + \frac{i}{2} [\psi^* (\partial_t \psi) - (\partial_t \psi^*) \psi] - V \psi^* \psi. \]

and find

\[ T^{0\mu} = -\frac{i}{2} [(\partial^\mu \psi^*) \psi - \psi^* (\partial^\mu \psi)] = \frac{i}{2} [\psi^* \overleftarrow{\partial^\mu} \psi] = -\rho \partial^\mu S \]

Recalling that

\[ P_B = \nabla S \quad \text{and} \quad E_B = -\partial_t S \]

Then we find

\[ \rho P_{jB} = \rho \partial_j S = -T^{0j} = -\frac{\rho}{2} [P_{jW}]_+ \]

\[ \rho E_B = -\rho \partial_t S = -T^{00} = -\frac{\rho}{2} [P_{tW}]_+ \]

This generalises to the Pauli and Dirac particles

The Bohm momentum.

The Bohm energy.

\[ \frac{1}{2} [P_{W}^2]_+ = (\nabla_x S(x))^2 - \frac{\nabla_x^2 R(x)}{R(x)} = P_B^2 + Q. \]

Quantum potential

\[ \frac{1}{2i} [P_{W}^2]_- = \nabla_x^2 S(x) + \left( \frac{\nabla_x \rho(x)}{\rho(x)} \right) \nabla_x S(x). \]


Weak values from Moyal algebra.

Multiplication of phase space functions defined by

\[ a(x, p) \star b(x, p) \quad \text{where} \quad \star = \exp \left[ \frac{i}{2} \left( \partial_x \partial_p - \partial_p \partial_x \right) \right] \]

Let \( f(x, p) \) be the density matrix in \((x, p)\) representation, viz.

\[ f(x, p) = \int \rho(x - y/2; x + y/2) e^{-ipy} dy \]

\[ p \star f(x, p) = \left( p - \frac{i}{2} \partial_x \right) f(x, p) \quad \text{and} \quad f(x, p) \star p = f(x, p) \left( p + \frac{i}{2} \partial_x \right) \]

Now form

\[ [p, f(x, p)]_{BB} := \frac{(f \star p + p \star f)}{2} = pf(x, p) \quad \text{Baker bracket} \]

\[ [p, f(x, p)]_{MB} := \frac{(f \star p - p \star f)}{i} = \nabla_x f(x, p) \quad \text{Moyal bracket} \]

\[ [x, p]_{MB} = 1 \]

To make contact with configuration space we must form a marginal;

\[ \int [p, f(x, p)]_{BB} dp = \int pf(x, p) dp = \rho(x) \overline{p}(x) \quad \text{Moyal momentum} \]

Using

\[ \rho(x) \overline{p}^n(x) = \left( \frac{1}{2i} \right)^n \left[ \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^n \psi(x_1) \psi^*(x_2) \right]_{x_1 = x_2 = x} \quad \text{(A1.1)} \]

we find \[ \overline{p}(x) = \nabla_x S(x) \quad \text{(A1.6)} \]

\[ P_B(x) = \int [p, f(x, p)]_{BB} dp / \rho(x) \quad \text{Bohm Momentum} \]

Finally

\[ \frac{\nabla_x \rho(x)}{2\rho(x)} = \left[ \int [p, f(x, p)]_{MB} dp \right] / \rho(x) \quad \text{Osmotic Momentum} \]
Kinetic energy.

Form
\[ p^2 \ast f(x, p) = [p^2 - ip \nabla_x - \frac{1}{4} \nabla_x^2] f(x, p). \quad \text{and} \quad f(x, p) \ast p^2 = f(x, p) [p^2 + ip \nabla_x - \frac{1}{4} \nabla_x^2] \]

Now form the Baker bracket
\[ [p^2, f(x, p)]_{BB} = p^2 f(x, p) - \frac{1}{4} \nabla_x^2 f(x, p). \]

We need to find the marginal
\[ \int [p^2, f(x, p)]_{BB} dp = \rho(x) p^2(x) - \frac{1}{4} \nabla_x^2 \rho(x). \]

with
\[ \overline{p^2}(x) = (\nabla_x S(x))^2 + \frac{1}{4} \frac{\nabla_x^2 \rho(x)}{\rho(x)} - \frac{\nabla_x^2 R(x)}{R(x)} \]

So that
\[ \int [p^2, f(x, p)]_{BB} dp / \rho(x) = (\nabla_x S(x))^2 - \nabla_x^2 R(x) / R(x) = P_B^2 + Q. \quad \text{[Remember } m=1/2.] \]

To complete the story, the Moyal bracket gives
\[ \int [p^2, F(x, p)]_{MB} dp / \rho(x) = \nabla_x^2 S(x) + \left( \frac{\nabla_x \rho(x)}{\rho(x)} \right) \nabla_x S(x). \]

Summary so far.

We are interested in the values that can be found experimentally using weak measurements.

We have seen how these weak values are related to the Bohm momentum, Bohm energy and the quantum potential

\[ \rho P_j B = \rho \partial_j S = -T^{0j}, \quad \rho E_B = -\rho \partial_t S = -T^{00} \]

The separation of the real from the imaginary parts of weak values achieved by forming brackets

\[ [[P_W]]_\pm = \left[ \frac{\langle \psi(t) | \vec{P} | x \rangle}{\langle \psi(t) | x \rangle} \pm \frac{\langle x | \vec{P} | \psi(t) \rangle}{\langle x | \psi(t) \rangle} \right] \]

Then, for example

\[ P_B = -\frac{1}{2} [[P_{xW}]]_+ \quad \text{and} \quad E_B = -\frac{1}{2} [[P_{tW}]]_+ \]

In the Moyal algebra these brackets are replaced by the Baker and Moyal brackets

\[ [[P_W]]_+ \Rightarrow [p, f(x, p)]_{BB} = \frac{(f \ast p + p \ast f)}{2} \quad \text{and} \quad [[P_W]]_- \Rightarrow [p, f(x, p)]_{MB} = \frac{(f \ast p - p \ast f)}{i} \]

What about spin and relativity?
Spin and Clifford Algebras.

We could use standard approach with matrices. Neater to use Clifford algebras.

One single mathematical structure with a natural hierarchy

One single mathematical structure with a natural hierarchy

Generating elements.

Conformal \( \{1, e_0, e_1, e_2, e_3, e_4, e_5\} \)

Twistors \( \{\omega, \pi\} \)

Dirac \( \{1, e_0, e_1, e_2, e_3\} \)

\( \{1, \gamma_0, \gamma_1, \gamma_2, \gamma_3\} \)

Pauli \( \{1, e_1, e_2, e_3\} \)

\( \{1, \sigma_1, \sigma_2, \sigma_3\} \)

Schrödinger \( \{1, e_1\} \)

Klein-Gordon. \( \{1, i\} \)

General element of algebra

\( A(x) = \sum g(x) e_K \)

where \( e_K = e_i \circ e_j \circ \cdots \circ e_n \) and \( i < j < \cdots < n \)

Clifford product

\( e_i \circ e_j + e_j \circ e_i = 2g_{ij} \)

Replacement of kets and bras by elements of the algebra.

Replace $\langle x|\psi \rangle$ by $\Phi_L(x) = \phi_L(x)\epsilon$

Replace $\langle \psi|x \rangle$ by $\widetilde{\Phi}_L(x) = \epsilon\tilde{\phi}_L(x)$

element of min left ideal

idempotent $\epsilon^2 = \epsilon$

Clifford reversion

Pauli

$\langle x|\Psi_P \rangle \rightarrow \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$

$\Phi_L(x) = \left[ g_0(x) + \sum g_i(x)e_{jk} \right] \epsilon$

$i, j, k$ cyclic

$g \in \mathbb{R}$

$2g_0 = (\psi_1^* + \psi_1)$

$2e_{123}g_3 = (\psi_1^* - \psi_1)$

$\epsilon = (1 + e_3)/2$

$2g_2 = (\psi_2^* + \psi_2)$

$2e_{123}g_1 = (\psi_2^* - \psi_2)$

Dirac

$\langle x|\Psi_D \rangle \rightarrow \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$

$\Phi_L(x) = \left[ g_0(x) + \sum g_i(x)e_{\mu\nu} + g_5(x)e_5 \right] \epsilon'$

$\mu < \nu$

$\epsilon' = (1 + \gamma_0)/2$

$\psi_1 = g_0 - ib$; $\psi_2 = -d - ic$; $\psi_3 = h - ig_5$; $\psi_4 = f + ig$

Complete by adding Schrödinger

$\langle x|\Psi_s \rangle \rightarrow \psi(x)$

$\Phi_L(x) = \left[ g_0(x) + g_1(x)e \right]$

$\epsilon = 1$

$2g_0 = (\psi^* + \psi)$

$2eg_1 = (\psi^* - \psi)$

Bohm momentum and energy for Pauli particle.

\[ P_B = -\frac{1}{2} [P_W]_+ \] now becomes

\[ \rho P_B(x) = -i \Phi_L(x) \nabla_x \bar{\Phi}_L(x) = -i \left[ (\nabla_x \Phi_L(x)) \bar{\Phi}_L(x) - \Phi_L(x) \left( \nabla_x \bar{\Phi}_L(x) \right) \right] \]

\[ \Phi_L(x) = \phi_L(x) \epsilon \] we choose the idempotent \( \epsilon = (1 + e_3)/2 \)

Pauli current = convection part + rotation part

The Bohm momentum comes from the convection part using \( \Phi_L(x) = \left[ g_0(x) + \sum g_i(x)e_{jk} \right] \epsilon \)

\[ \rho P_B = -e_{123}[g_0 \nabla_x g_3 - g_3 \nabla_x g_0 + g_2 \nabla_x g_1 - g_1 \nabla_x g_2]. \]

Using the conversion \( g(x) \rightarrow \psi(x) \) and \( \psi_j(x) = R_j(x)e^{iS_j(x)} \)

\[ \rho P_B(x) = \rho_1(x) \nabla_x S_1(x) + \rho_2(x) \nabla_x S_2(x) = -\frac{1}{2} [P_W]_+ \]

Bohm Momentum

\[ \rho E_B(x) = \rho_1(x) \partial_t S_1(x) + \rho_2(x) \partial_t S_2(x) = -\frac{1}{2} [P_{tW}]_+ \]

Bohm Energy

Bohm, Schiller and Tiomno spin-1/2 model.

\[ P_B = (\nabla S + \cos \theta \nabla \phi)/2 \quad E_B = -(\partial_t S + \cos \theta \partial_t \phi)/2 \]

They are the same if you use

\[ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \exp(i\phi/2) \\ i \sin(\theta/2) \exp(-i\phi/2) \end{pmatrix} \exp(i\psi/2) \]


Thursday, 2 February 2012
Bohm kinetic energy for spin-1/2 particle.

Then as shown in Hiley and Callaghan, the Bohm kinetic energy is

\[
- m [P_W^2]_+ = P_B^2(x) + [2(\nabla_x W(x) \cdot S(x)) + W^2(x)] = P_B^2 + Q.
\]

Spin of particle \( S = i(\phi_L e_3 \bar{\phi}_L) \) and \( \rho W = \nabla_x (\rho S) \)


BST values:

\[
Q = -\frac{1}{2m} \nabla^2 R + \left[ (\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2 \right] / 8m
\]


Thursday, 2 February 2012
Spin in the Moyal Algebra.

Central to the Clifford algebra approach is the Clifford density element.

\[ \rho(x) = \Phi_L(x) \tilde{\Phi}_L(x) \]

Then

\[ \langle A \rangle = Tr[A\rho(x)] = Tr[A\Phi_L(x)\tilde{\Phi}_L(x)] = Tr[\Phi_L(x)A\tilde{\Phi}_L(x)] = \langle \psi | \hat{A} | \psi \rangle \]

Generalise

\[ \rho(x_1, x_2) = \Phi_L(x_1) \Xi_R(x_2) \]

Now we are in a position to extend this to phase space using a Wigner-Moyal transformation

\[ \rho_M(x, p) = F(x, p) = 2\pi \int \Phi_L(x_1) \Xi_R(x_2) e^{ipy} dy \]

Cross-Wigner function.

where \( x = (x_2 + x_1)/2 \) and \( y = x_2 - x_1 \)

Recall for Pauli

\[ \Phi_L(x) = \left[ g_0(x) + \sum g_i(x)e_{jk} \right] \epsilon \quad \text{Replaces spinor ket.} \]

\[ \Xi_R(x) = \left[ G_0(x) - \sum G_i(x)e_{jk} \right] \epsilon \quad \text{Replaces spinor bra.} \]

Use these expressions in the cross-Wigner function.
Details of Spin in the Moyal Algebra.

Easier to use the momentum representation. \[ F(x, p) = \left( \frac{1}{2\pi} \right) \int \Xi_L(p_1) \tilde{\Xi}_L(p_2) e^{-i x \Delta p} d\Delta p. \]

Now with \[ p = (p_2 + p_1)/2 \] and \[ \Delta p = p_2 - p_1 \]

and \[ \Xi_L(p) = \xi(p)\epsilon = \left[ \gamma_0(p) + \sum \gamma_K(p)e_K \right] \epsilon, \] where the \( \gamma(p) \)s are Fourier transforms of the \( g(x) \)s.

Again choose \( \epsilon = (1 + e_3)/2 \), taking just the 1 term and then define \[ 2F(x, p) := \int \xi_L(p_2) \tilde{\xi}_L(p_1) e^{-i x \Delta p} d\Delta p \]

So that \[ 2F(x, p) = \int [\gamma_0(p_2)\gamma_0(p_1) + \gamma_1(p_2)\gamma_1(p_1) + \gamma_2(p_2)\gamma_2(p_1) + \gamma_3(p_2)\gamma_3(p_1)]e^{-i x \Delta p} d\Delta p. \]

Now form \[ p \star F(x, p) = pF(x, p) - \frac{i}{2} \nabla_x F(x, p) \quad \text{and} \quad F(x, p) \star p = pF(x, p) + \frac{i}{2} \nabla_x F(x, p) \]

**Baker bracket** \[ [p, F]_{BB} = pF(x, p), \]

**Moyal bracket** \[ [p, F]_{MB} = \nabla_x F(x, p). \]

Use the Moyal relation

\[ F(x, p) = \frac{1}{2\pi} \iiint M(p_1, p_2) \delta \left( p - \frac{p_2 - p_1}{2} \right) e^{i x (p_2 - p_1)} dp_1 dp_2 \]

where \[ M(p_1, p_2) = \frac{1}{2} [\phi_0(p_1)\phi_0^*(p_2) + \phi_1(p_1)\phi_1^*(p_2) + \phi_2(p_1)\phi_2^*(p_2) + \phi_3(p_1)\phi_3^*(p_2)] \]

so that we find

\[ \int [p, F]_{BB} dp = \rho_1(x)\partial_x S_1(x) + \rho_2(x)\partial_x S_2(x). \quad \text{Baker bracket} \leftrightarrow \text{Bohm momentum.} \]

\[ \int [p^2, F(x, p)]_{BB} dp = \rho_1(x)(\nabla_x S_1(x))^2 + \rho_2(x)(\nabla_x S_2(x))^2 - R_1(x)\nabla_x^2 R_1(x) - R_2(x)\nabla_x^2 R. \]

**Bohm KE**

**Quantum Potential**
Time Development Equations.

\( H(x, p) \star f(x, p) = E_1 f(x, p) \) or \( f(x, p) \star H(x, p) = E_2 f(x, p) \)

Time development

\[
H \star f = \frac{i}{2\pi} \int \psi^*(x_2) [\partial_t \psi(x_1)] e^{ipy} dy \\
f \star H = \frac{-i}{2\pi} \int [\partial_t \psi^*(x_2)] \psi(x_1) e^{ipy} dy
\]

Difference

\[
\frac{\partial f}{\partial t} + [f, H]_{MB} = 0 \quad \text{In the limit } O(\hbar) \rightarrow \text{ Classical Liouville eqn.}
\]

Sum

\[
\frac{i}{\pi} \int \psi^*(x_2) \partial_t \psi(x_1) e^{ipy} dy + [f, H]_{BB} = 0
\]

\[
f(x, p) \mathcal{E}_B(x, p) = \frac{i}{2\pi} \int \psi^*(x_2) \partial_t \psi(x_1) e^{ipy} dy
\]

In the limit \( O(\hbar) \rightarrow \quad \frac{\partial S}{\partial t} + H = 0 \quad \text{Hamilton-Jacobi eqn.} \)

Take the marginal as before

\[
\frac{\partial f(x, p)}{\partial t} + [f, H]_{MB} = 0 \quad \text{is equivalent to} \quad \frac{\partial \rho(x)}{\partial t} + [\rho, H]_+ = 0 \quad \text{Quantum Liouville eqn.}
\]

\[
2f(x, p) \mathcal{E}_B(x, p) + [f, H]_{BB} = 0 \quad \text{is equivalent to} \quad 2\rho(x) E_B(x) + [\rho, H]_+ = 0
\]

If \( H = \frac{p^2(x)}{2m} + V(x) \) then \( \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0 \quad \text{Quantum Hamilton-Jacobi eqn.} \)
Summary of Algebraic Time Evolution Equations.

\[ \frac{\partial F}{\partial t} + [F, H]_{MB} = 0 \]
\[ i \frac{\partial \rho}{\partial t} + [\rho, H]_- = 0 \]
\[ 2 \frac{\partial S}{\partial t} F + [F, H]_{BB} = 0 \]
\[ 2 \frac{\partial S}{\partial t} \rho + [\rho, H]_+ = 0 \]

**Moyal algebra**
Phase space

**Quantum algebra**
Configuration space

Notice in the general form there is no quantum potential.

The QP appears ONLY in a representation

In the \(x\)-representation you get Bohm’s \(Q(x)\).

In the \(p\)-representation you get another QP--\(Q(p)\)

In Moyal take the \(x\)-marginal.

Mathematical Structure.

Orthogonal Clifford + Moyal

Symplectic Clifford.

Take marginals

Configuration space
  Position

Configuration space
  Momentum

Order $\hbar$

Classical phase space

[Crumeyrolle Orthogonal and Symplectic Clifford Algebras, (1990)]
Physics behind these Algebras?

Basic structure

\[ \rho_{\phi \psi}(x_1, x_2) \rightarrow f(x, p) \]

Transition \(\Rightarrow\) Process

“Phase” space

\[ \rho_M(x, p) = 2\pi \int \Phi_L(x_1) \Xi_R(x_2) e^{ipy} dy \]

Quantum particle not a ‘rock’ but a ‘blob’.

Structure process.

[Bohm, Proc. Int. Conf. on Elementary Particles, Kyoto, 252-287, (1965)]

Only combine if \( t = s' \)

Orthogonal and symplectic product \(\Rightarrow\) order of succession and the order of co-existence \(\Rightarrow\) addition gives the algebra.

Combine Clifford and Moyal algebras \(\Rightarrow\) Non-commutative geometry

Orthogonal Clifford and symplectic Clifford algebras.


[Lizzi, Non-commutative spaces, Springer Lecture Notes in Physics 774, 2009]
Consequences of Non-commutative Structure.

Changes from both sides  

Inner automorphism \( TAT^{-1} \)

Evolution in \( \tau \)  
\[ A_\tau = T(\tau)A_{\tau_0}T(\tau)^{-1} \]

If \( T = \exp[iH\tau] \) then for small \( \tau \)

\[ i\left( \frac{A_\tau - A_0}{\tau} \right) = (HA_0 - A_0H) \quad \Rightarrow \quad i\dot{A} = [A, H] \]

Heisenberg eqn.

Just Bohm’s ‘folding’ and ‘unfolding’.

\[ [T_1, T_2] \]

Overarching Philosophy.


The deeper structure gives rise to a non-commutative phase space geometry.

In this context non-commutativity $\Rightarrow$ not all orders can be made explicit together.

Implicate order $\Rightarrow$ algebra

Project out explicate orders

$\psi(x)$ or $\phi(p)$

Quantum world

Classical world

Thursday, 2 February 2012