THIRD PROLOGUE Coherent Radiation of an Ultra-Relativistic Charged Particle in a Medium

This example of qualitative reasoning is borrowed from the paper by V.M. Galitskii and I.I. Gurevich [1][1]. Following this paper, we use the quantum relativistic units for which $c = \hbar = 1$. An important dimensionless parameter of the problem is the Lorentz factor $\gamma = E/m$, which we assumed of being large $\gamma \gg 1$.

Coherent Radiation, Radiation Length and Reduction Factor

When a fast particle is passing through a medium, it is scattered by constituting it atoms. This scattering is accompanied by a transfer of energy and momentum from fast

[1] Another and, may be, easier accessible source is presented by the review article by M.I. Ryazanov [2]

particle to the atoms. The momentum transfer leads to the radiation (*Bremsstrahlung*). Typical momentum transfer Δp is limited by inverse size $1/a_B$ of the atom and, for fast incident particles, is small compared to momentum E of the particle ($\Delta p \sim 1/a_B \ll p \approx E$). This leads to the scattering to a small angle

$$\alpha \sim \frac{1}{Ea_B} \tag{1}$$

So, the track of fast particle is nearly a strait line. If the fast particle moves with the speed nearly equal to speed of light, and the radiation is directed at a small angle θ to the particle's trajectory, then there is a long distance l at which the particle and light quanta propagate with nearly equal speed and the processes of light radiation at the frequency ω are coherent. So, the intensity of radiation $I(\theta, \omega)$ is

$$I(\theta, \omega) = A\ l^2(\theta, \omega), \tag{2}$$

where A denotes a coefficient independent on radiation angle θ. For finding the coherence length l_0, look at the sketch in Fig 1. and compare time of the particle passing with ve-

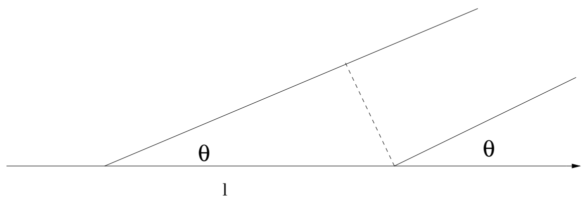

Figure 1: The sketch of the coherent light emition under the angle θ to the trajectory of the fast particle

locity v along l_0 and light passing along $l_0 \cos\theta$

$$\frac{l_0}{v} - l_0 \cos\theta = \lambda \sim \frac{1}{\omega}, \qquad l_0(\omega,\theta) \sim \frac{v}{\omega\ (1 - v\cos\theta)} \tag{3}$$

Since the velocity v is coupled with energy E as

$$v = 1 - \frac{1}{\gamma^2} \tag{4}$$

and

$$l_0(\omega,\theta) \sim \frac{1}{\omega\,[1 - \ (1-\gamma^{-2})\cos\theta]}, \quad l_0(\omega,0) \sim \frac{\gamma^2}{\omega} \tag{5}$$

It follows from Eq (5) that characteristic angle $\tilde{\theta}$ of radiation is

$$\tilde{\theta} \sim \frac{1}{\gamma}$$

and the intensity of radiation $I(\omega)$ at frequency ω is:

$$I(\omega) \approx 2\pi A(\omega) \int \theta\ d\ \theta\ l^2(\omega,\theta) \sim \pi A(\omega)\ \frac{\gamma^2}{\omega^2} \tag{6}$$

All mentioned dependences are the same as in the exact Bethe-Heitler formula [3]

$$I_{BH}(\omega) \ = \ \frac{e^2}{3\pi}\ \frac{E_s^2}{m^2 L}, \tag{7}$$

$$E_s = m\ \sqrt{4\pi \cdot 137}, \qquad L^{-1} = 4ne^2 \left(\frac{Ze^2}{m}\right)^2 \ln\left(\frac{191}{Z^{1/3}}\right)$$

This gives for parameter A in Eq (3)

$$A = \frac{1}{\pi}\ \left(\frac{m\omega}{2\pi E}\right)^2\ I_{BH}(\omega) \tag{8}$$

Considering effects of a dense medium on the total rate of bremsstrahlung $I(\omega)$, obtain

$$I(\omega) = I_{BH}(\omega)\ \frac{l(\omega,0)}{l_0(\omega,0)} \tag{9}$$

where $l_0(\omega,0)$ is the coherent length for the light emition at zero angle (exactly forward). Therefore, considering effects of a dense medium on bremsstrahlung, we could concentrate on its effects on coherence length.

Effects of Medium Polarisation and Multiple Scattering

Two major factor of the dense medium are its polarisation and the multiple scattering of the fast particles. Due to multiple scattering to angle θ_s, the velocity v along the averaged direction of motion is renormalised as

$$v_{\|} = v\langle\cos\theta_s\rangle \approx v\left(1 - \frac{\langle\theta_s^2\rangle}{2}\right) \tag{10}$$

In what follows, we will need the mean square angle $\langle\theta_s^2\rangle$ at the length l. It is equal to

$$\langle\theta_s^2\rangle = l\ \frac{E_s^2}{E^2\ L} \tag{11}$$

Medium polarisation could be taken into account through effective dielectric permeability

$$\epsilon(\omega) = 1 - \frac{\Omega^2}{\omega^2}, \quad \Omega^2 = \frac{4\pi Zne^2}{m_e} \tag{12}$$

Since $\epsilon < 1$, light propagate in medium factor than in vacuum and easier to get "detached" from the moving particle.

Taking into account both two factors, obtain for the correlation length

$$l(\omega,\theta) = \frac{\lambda}{2}\left[\epsilon^{-1/2} - v(1-\cos\theta)\left(1 - \frac{\langle\theta_s^2\rangle}{2}\right)\right]^{-1} \quad (13)$$

Substituting expressions (11) and (12) into Eq (13) and taking into account the inequalities $E \gg m$ and $\omega \gg \Omega$, obtain

$$l(\omega,0) = \left[\frac{m^2}{E^2}\, m + \frac{\Omega^2}{\omega^2} + \frac{E_s^2}{E^2}\frac{\omega}{L}\, l(\omega,0)\right]^{-1} \quad (14)$$

Thus, we obtain a quadratic equation for coherence length $l(\omega,0)$. Solving it, obtain

$$l_{\omega,0} = \frac{Lm^2}{2E_s^2}\left[\sqrt{\left(1+\frac{E^2\Omega^2}{m^2\omega^2}\right)^2 + 4\frac{E^2E_s^2}{m^4\omega L}} - 1 - \frac{E^2\Omega^2}{m^2\omega^2}\right] \quad (15)$$

Eq (15) allows significant simplification if

$$\frac{E}{m}\frac{\Omega}{\omega} \ll 1, \qquad \frac{m^2}{E^2} \gg \frac{E_s^2}{m^2\ \omega\ L} \equiv \frac{E_s^2}{E^2}\frac{l_0(\omega,0)}{L} \quad (16)$$

The second of these inequalities means that the mean angle θ_s of multiple scattering is less, than the typical angle $\theta \sim m/E$ of radiation. Under these conditions

$$l(\omega,0) \approx \frac{E^2}{m^2\ \omega} \equiv l_0(\omega,0) \quad (17)$$

and the spectrum of radiation coincides with that by Bethe and Heitler (7). If the medium polarisation is weak but the effect of the scattering is strong

$$1 \ll \left(\frac{E\ \Omega}{m\ \omega}\right)^2 \ll \frac{E^2\ E_s^2}{m^4\ \omega\ L}, \tag{18}$$

then the coherence length $l(\omega, 0)$ take the following form

$$l(\omega, 0) = \frac{E}{E_s}\ \sqrt{\frac{L}{\omega}} \tag{19}$$

and the spectrum of radiation is

$$I(\omega) \approx \frac{e^2}{3\pi}\ \frac{E_s}{E}\ \sqrt{\frac{L}{\omega}} \tag{20}$$

One could see from Eq (20) that, in a contrast with BH formula, the spectrum of radiation $I(\omega)$ vanishes at $\omega \to 0$. This spectrum was first obtained by Landau and Pomeranchuk [4]. It follows from Eq (18) that the multiple scattering leads to strong effects if

$$E > m\ \left(\frac{m}{E_s}\right)^2\ L\ \Omega \tag{21}$$

In the limiting case

$$1 \ll \frac{E^2\ E_s^2}{m^4\ \omega\ L} \ll \left(\frac{E\ \Omega}{m\ \omega}\right)^2, \tag{22}$$

opposite to that of Eq (18), polarisation plays leading role in suppression of coherence length $l(\omega, 0)$ which takes now the following simple form :

$$l(\omega, 0) = \frac{\omega}{\Omega^2} \tag{23}$$

The bremsstrahlung has in this case the following spectrum of radiation

$$I(\omega) \approx \frac{e^2}{3\pi\ L}\ \left(\frac{E_s\ \omega}{E\ \Omega}\right)^2 \tag{24}$$

first found by Ter-Mikaelyan [5]

Effects of Light Absorption

One more phenomenon which could change the frequency spectrum and the rate of Bremsstrahlung is light absorption due to pair creation. It becomes relevant if the correlation length of Bremsstrahlung l exceeds the radiation length L ($l \geq L$). Under this condition for the purposes of an estimate, the correlation length $l(\omega, 0)$ must be replaced by the radiation length L. This gives condition

$$E > E_c = 2E_S\ L\ \Omega \tag{25}$$

under which the absorption is essential. For frequency range

$$L\ \Omega^2 < \omega < \omega_c = \frac{E^2}{E_s^2\ L} \tag{26}$$

the spectrum of radiation is

$$I_c(\omega) \approx \frac{l^2}{3\pi}\ \left(\frac{E_s}{E}\right)^2\ \omega \tag{27}$$

Conclusion.
Map of “The World of Parameters”

In conclusion, consider the Map of “The World of Parameters” for Bremsstrahlung in a dense medium. This map is presented in Fig 2.

The region I is bounded by two straight lines: OB ($\omega = E$) and OD ($\omega = E\,\Omega/m$) - and the arch BD ($\omega = E^2 E_s^2/m^4 L$). Inside this region, all effects of a dense medium are negligible and the Bethe-Heitler formula for intensity of Bremsstrahlung is valid.

In region II below line OC ($\omega \ll E\ \Omega/m$), the leading factor is the medium polarisability and the Ter-Mikaelyan formula Eq (24) is valid.

In region III between two arches BD ($\omega = E^2 E_s^2/m^4 L$)

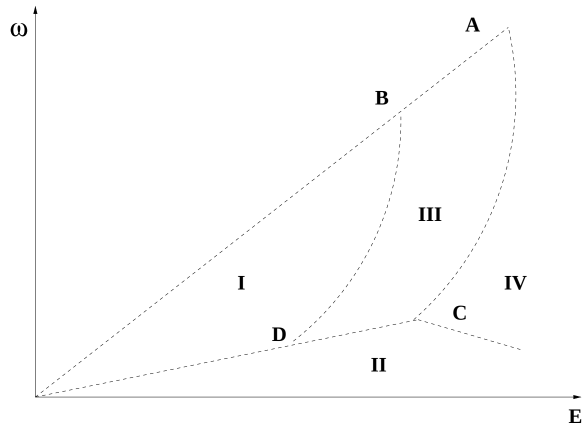

Figure 2: Map of “The World of Parameters”: E is the energy and ω is the frequency of bremsstrahlung radiation. The regions I - IV correspond to the limiting cases (see in the text).

and AC ($\omega = E^2/E_s^2\ L$), the leading factor is the multiple scattering of radiating particle, and the Landau-Pomeranchuk formula Eq (20) is valid.

Finally, in region IV, the leading factor is the light absorption and the formula (27) is valid.

A special exercise is to find out that the asymptotic expressions for intensity of Bremsstrahlung are matching at the boundary between the relevant regions[2]. In case the asymptotes are not matching means that there is an intermediate region with an intermediate asymptote.

[2] They are matching in our case

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