Tale 56
Frobenius' Numbers and Quantum Mechanics

Numbers

We need numbers at any event and for multiple purposes. In different case we use very different numbers. Let us discuss which properties we want the numbers to have. First of all, we want to have an opportunity to add two numbers, so the summ $c = a + b$ of two numbers a and b is a number itself. We also want to have an opportunity to multiply two numbers a and b, so their product $c = a \cdot b$ is a number itself.

Let us now discuss the properties of these two operations. First of all, we want the addition to be commutatve and to form a group operation. This means that:

1. $a + b = b + a$
2. There is a zero element 0, so, for any element a
   $a + 0 = 0 + a = a$

3. For any element a, there is its negative image $-a$, so $-a + a = a + (-a) = 0$

4. All numbers obey the associativeness law: $(a + b) + c = a + (b + c)$

   One more thing is that we demand the multiplication to be a group operation as well:

5. There is unity element 1, so, for any element a $a \cdot 1 = 1 \cdot a = a$

6. For any element a but zero 0, there is its inverted image a^{-1}, so $a^{-1} \cdot a = a \cdot a^{-1} = 1$

7. All numbers obey the associativeness law: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

   Finally, we want multiplication to be distributive with respect to addition:

8. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Any set of elments F, on which two operations are determined obeying the rules 1 - 8, is called the *field*. Any field contains two elements: 0 and 1. Therefore, the minimal field $\mathbf{Z_2}$ consists of only these two elements with the rules of composition:

$$0 + 0 = 0 \quad 0 + 1 = 1 \quad 1 + 1 = 0$$

$$0 \cdot 0 = 0 \quad 0 \cdot 1 = 0 \quad 1 \cdot 1 = 1$$

Apparently, the set **Z** of all integer numbers - positive and negatve - forms a field. Thera many others. Say, the numbers of the form $x = a + b\sqrt{2}$, where a and b are real numbers, and operation of addition $x_1 + x_2$ and multiplication $x_1 \cdot x_2$ are like those for conventional real numbers, form another field.

Finally, the set of all real numbers **R** forms the field. Important extension of the field of real numbers is the field **C** of complex numbers $z = x + iy$, where x and y are real numbers and i is the imaginary unity ($i^2 = -1$). We all know why intruduction of complex number was necessary: polynomials with real coefficients not always have roots in real numbers, while always have complex roots. If the coefficients of polynomials are complex, then the roots still could be found among these numbers. These statement is expressed in sayng that unlike the field of real numbers, that of comlex numbers is algebraically closed.

An important new operation in the field **C** is complex conjugation (if $z = x + iy$, then $\bar{z} = x - iy$). A modulos of a complex number $|z|$ is determined by indentity

$$|z|^2 = \bar{z}z = x^2 + y^2$$

Using this operation, inversion could be written in the following form :

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\bar{z} \cdot z} = \frac{\bar{z}}{|z|^2}.$$

Is there any further extension of the field of real numbers ? A celebrated theorem by Frobenius says that, if we assume that multiplication commutative, then there is no further extension. For the case of non-commutative multiplication, there is only one exstension: the field of real quaternions **Q**. Similar to complex numbers, the quaternion q could be presented by its projection to four quaternion units:

$$q = a_0\ \mathbf{1} + a_1\ \tau_1 + a_2\ \tau_2 + a_3\ \tau_3,$$

where a_i are all real numbers, $\mathbf{1}$ is unity of multiplication and multiplication of all other units obeys the following rules:

$$\tau_i \cdot \tau_j = -\delta_{ij} + e_{ijk}\tau_k.$$

It is easier to see many properties of quaternions keeping in mind their representatation by 2×2 matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \tau_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\ \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ \tau_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

One can see that relation between quaternion units and the Pauli matrices $\tau_k = i\sigma^k$. All τ-matrices are anti-hermitian and anti-commutating. One can establish several discrete operations over quaternions:

1. Trace operation tr $q = a_0\mathbf{1}$ which projects the quaternion to the $\mathbf{1}$ axis.

2. Complex conjugation. If

   $$q = a_0\mathbf{1} + a_1\tau_1 + a_2\tau_2 + a_3\tau_3,$$

   then

   $$\bar{q} = a_0\mathbf{1} - a_1\tau_1 + a_2\tau_2 - a_3\tau_3.$$

3. Transposition.

   $$q^T = a_0\mathbf{1} + a_1\tau_1 - a_2\tau_2 + a_3\tau_3$$

4. Hermitian conjugation

$$q^{+} = (\bar{q})^{T} = a_0\mathbf{1} - a_1\tau_1 - a_2\tau_2 - a_3\tau_3$$

Modulus of a quaternion $|q|$ is a positive real number

$$|q|^2 = \bar{q} \cdot q$$

Therefore,

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

Thus, three Frobenius' fields **R**, **C** and **Q** form the basis of Calculus we are going to develope in following sections. One more thing we will come across is invariance in of this calculus with respect to certain contineous transformation. These transformations form the groups of transformations, and generators of these groups form the algebras. Elements of these algebras - we will see this in a diffferent fairy tale - could be real numbers, complex number and quaternions. It turns out that they also could octanions, discussed in the Appendix to this Fairy Tale.

Quantum Mechanics and Level Repulsion

Consider a quantum random system. The simplest example is presented by a single quantum particle in a box of a random shape and/or with a random background potential. Under this condition, the energy levels are also the random values. If the levels with large quantum numbers are concerned, the distribution function $P(E, E+\omega)$ for two levels to have the energies E and $E+\omega$, in the leading approximation, does not depend on E:

$$P(E, E+\omega) = P(\omega)$$

The simplest question is about general properties of this function. Characteristic energy is the level spacing $\Delta = 1/\nu V$, where V is the volume of the box and ν is the density of states at energy $E \gg \Delta$. Therefore, the level statistics is determines by the dimensionless function $P(\omega) = P(\omega/\Delta)$ of dimensionless argument $x = \omega/\Delta$. The question is what are the properties of this function. In particular, what are its asymptotes at $x \ll 1$ and $x \gg 1$.

Wigner Ensembles

With all the generality of the problem, we must convince ourself whether the Pair correlation function $P(x)$ is universal, i.e. it is the same for all kind of random systems. Very soon, we will see that there are three different Ensembles of random systems. So, $P(x)$ is universal within each Ensemble and differs for different ensembles. Eugene Wigner was first, who addressed this question and came to conclusion about origin of distinction between the ensembles.

O. First of all, the systems with the real Hamiltonians have real eigen-functions. In any basis, formed by real functions $\psi_n(\mathbf{r})$, such a Hamiltonian is represented by a symmetric matrix with real matrix elements. A transformation from one basis of this type to another is made by orthogonal matrices. This is why, Wigner called random matrices of this type *an Orthogonal Ensemble.*

U. In case our box is position in a magnetic field, all eigenfunction $\psi_n(\mathbf{r})$ cannot be chosen real simultaneously. Some of them have to have a non-vanishing imaginary part. As the result, the Hamiltonians of such a system is presented by a hermitian matrix. A transformation from one basis of this type to another is made by unitary matrices. This is why the random matrices of this type belong to *a Unitary Ensemble.* Very similar effect makes any other part in the Hamiltonian, which breaks the symmetry in respect to the inversion. For instance, if the particle in the box is neutral, but has a non-zero magnetic moment associated with the spin - like, for instance, neutron - then magnetic field leads to Zeemann splitting of the energy levels and, together with spin-orbit interaction, this makes the Hamiltonian matrix hermitian and not real.

S. Finally, a very special situation occurs, when the particle in the box has a semi-integer spin, and the random potential has a spin-orbital component, while there is no magnetic field or any other part in the Hamiltonian which break the symmetry with respect to time inversion. In this case, all energy levels are presented by the Kramers' spin doublets

$$\psi(\mathbf{r}) = \begin{pmatrix} \phi_1(\mathbf{r}) \\ \phi_2(\mathbf{r}) \end{pmatrix}, \quad \psi_2^* = \psi_1. \tag{1}$$

The Hamiltonian of such a system is presented by the matrices, all elements of which are real quaterneons. A transformation from one basis of the type of presented by Eq (1) is made by symplectic matrices. This is why the random matrices of this type belong to *a Symplectic Ensamble*.

Pair Correlation Function $P(x)$ at $x \ll 1$

Consider now the asymptote of the Pair Correlation Function $P(x)$ at $x \ll 1$. In dimensional units, this inequality read $\omega \ll \Delta$, which means that the difference between the energies of two levels is much smaller, than that with the other levels. This allows to consider only these two levels.

O. An effective Hamiltonian $\hat{H}_O$ for two-level system of the Orthogonal Ensemble is presented by the real symmetric matrix of the following form

$$\hat{H}_O = \begin{pmatrix} \epsilon & V \\ V & -\epsilon \end{pmatrix}. \tag{2}$$

The eigen-energies $E_\pm$ for the matrix (2) and the energy difference $E_+ - E_-$ are

$$E_\pm = \pm\sqrt{\epsilon^2 + V^2}, \qquad E_+ - E_- = 2\,\sqrt{\epsilon^2 + V^2} \tag{3}$$

For small values of real parameters ϵ and V, the probability densities are independent of these values. Therefore, the pair correlation function $P_O(\omega)$ is proportional to

$$P_O(\omega) \propto \int d\epsilon\; dV\; \delta\left(\omega - 2\sqrt{\epsilon^2 + V^2}\right)$$
$$\propto \int_0^\infty \rho\; d\rho\; \delta(\omega - 2\rho) \propto \omega, \qquad P_O(x) \propto x \tag{4}$$

U. Repeating very same arguments in the case of the Unitary Ensemble, arrive at the Hamiltonian $\hat{H}_U$ in the form of the hermitian matrix

$$\hat{H}_U = \begin{pmatrix} \epsilon & V_1 + i\ V_2 \\ V_1 - i\ V_2 & -\epsilon \end{pmatrix}. \tag{5}$$

with eigenvalues

$$E_\pm = \pm\sqrt{\epsilon^2 + V_1^2 + V_2^2}, \quad E_+ - E_- = 2\ \sqrt{\epsilon^2 + V_1^2 + V_2^2} \tag{6}$$

and the Pair Correlation Function

$$P_U(\omega) \propto \int d\epsilon\ dV_1\ dV_2\ \delta\left(\omega - 2\sqrt{\epsilon^2 + V_1^2 + V_2^2}\right)$$
$$\propto \int_0^\infty \rho^2\ d\rho\ \delta(\omega - 2\rho) \propto \omega^2, \qquad P_U(x) \propto x^2 \tag{7}$$

S. In the case of Symplectic Ensemble, the inequality $\omega \ll \Delta$ means that two Kramers' doublets have very close energies. This means that the elements of the Hamiltonian matrix $\hat{H}_S$ are real quaterneons. Using Pauli matrices $\hat{\tau}^i$ to make a distinction between two doublets and $\hat{\sigma}^i$ for make a distinction between components of each doublet, the Hamiltonian matrix $\hat{H}_S$ could be presented in the following form

$$\hat{H}_S = \epsilon\ \tau^3 + V\ \tau^1 + i\ l^j\ \sigma^j\ \tau^2, \tag{8}$$

where all parameters ϵ, V and l^j are real. The eigenvalues of this matrix are

$$E_\pm = \pm\sqrt{\epsilon^2 + V + \mathbf{l}^2}, \quad E_+ - E_- = 2\ \sqrt{\epsilon^2 + V^2 + \mathbf{l}^2} \tag{9}$$

and the Pair Correlation Function

$$P_S(\omega) \propto \int d\epsilon\ d\ V\ d\mathbf{l}\ \delta\left(\omega - 2\sqrt{\epsilon^2 + V^2 + \mathbf{l}^2}\right)$$
$$\propto \int_0^\infty \rho^4\ d\rho\ \delta(\omega - 2\rho) \propto \omega^4, \qquad P_S(x) \propto x^4 \tag{10}$$

Probability $P(x)$ at $x \gg 1$

Eqs (4), (7) and (10) give the asymptotes of the Probability $P(x)$ to find two levels at dimensionless energy difference x, provided $x \ll 1$. At $x \gg 1$, $P(x) \to 1$ for all three ensembles. Dyson first pointed out that at $x \gg 1$ the difference $1 - P(x)$ decays as x^{-2} and oscillates. For instance

$$P_U(x) = 1 - \frac{\sin^2 x}{x^2} \tag{11}$$

The reason for such a slow decay of correlation is that the energy levels has a certain mean density and, therefore behave like an elastic chain.

Appendix. Octanions

If we sacrifise not only commutativeness of multiplication but also it associativeness, new numbers appear called *octanions* or *Calley numbers*. The set of octanions **O** with addition and multiplication forms new algebraic entity, called the *body*. It obeys all the axioms 1-8 the fields obey but that #7 of associativeness of multiplication.