Tale 15
A Climber and Geodesic Lines

Misha Spector told me this tale in 1970. Or rather, he suggested that I solved a problem, which I did, and we had a lively discussion. What I especially liked in this problem was its simple solution: you did not even need to write any equations. So, a mountaineer climbs an icy cone-shaped mountain. He has made a loop with a non-sliding knot and thrown it over the top of the mountain as shown in Fig 1. If the mountain is steep enough, the climber will reach the summit, if it is not, the loop will slip off the top. The problem is to find the critical

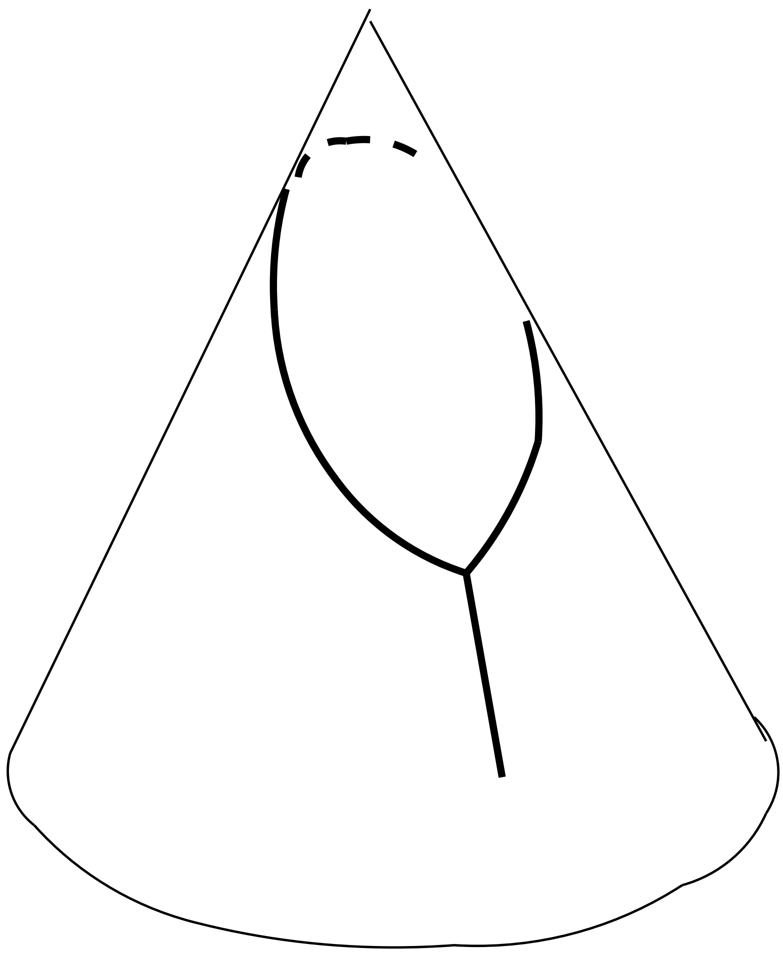

Figure 1: The rope on slop of icy mountain

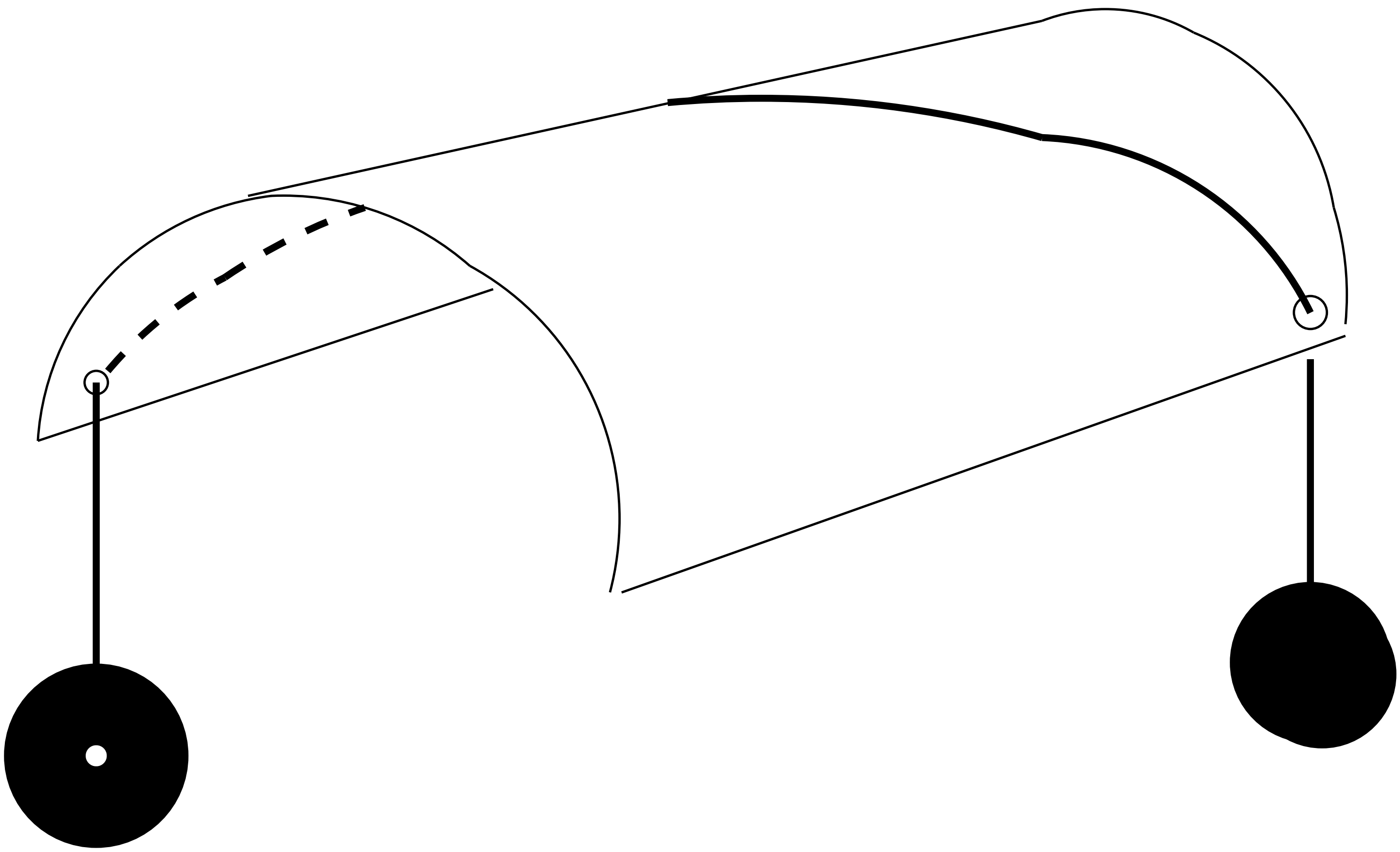

Figure 2: Lemma: the rope goes along a geodesic line

slope of the mountain.

The solution is given by a simple lemma, consisting in the following: Consider a rope AB on a convex surface with two equal weights at its ends (Fig. 2). It is obvious that the length AB must be the shortest possible, otherwise the potential energy of the weights would not be minimum. So, according to the lemma, a rope on a convex surface lies along a geodesic line, i.e. the shortest curve between two points lying wholly on the surface. Now, if we cut our cone along the generatrix, passing through the knot of the loop, and then roll it out on the plane, we will get a sector, shown in Fig. 3. Here A and B are the projections of the knot C on the plane. A geodesic on a plane is a straight line. So, the straight line AB in Fig. 3a is exactly what we need. Moreover,

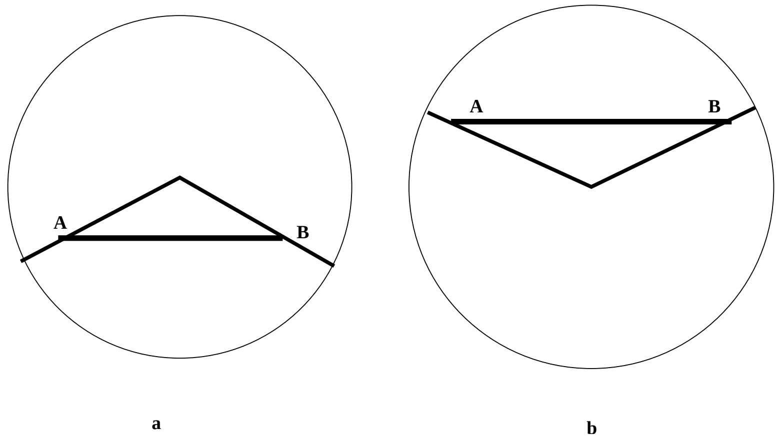

Figure 3: Open cone and the trace of the rope on it

we can see that the loop does not slip off the top of the mountain only if the angle θ does not exceed 180^o. Otherwise (see Fig. 3b), the line AB lies outside the sector and the loop has no equilibrium on the cone. To find the critical angle α of the cone, we note that if l is the generatrix of the cone, then the length θl of the arc, which subtends the angle θ, equals the circumference of the base of the cone $2\pi r$, where r is the radius of the base:

$$\theta l = 2\pi r.$$

As $r = l \sin\alpha$ and the critical angle $\theta = \pi$,

$$\sin\alpha = \frac{1}{2}, \qquad \alpha = 30^o$$