PROBLEMS ON PATH INTEGRALS

1. Calculate rate of pair production in external electric field *J.Schwinger, 1948*

2. Find conditions of quantization for Bloch electron with an arbitrary dispersion law in external magnetic field *L.Onsager, I.Lifshits, 1951*

3. Find distribution function of the winding numbers around the initial point of a random walk in two-dimension. The same for distribution function o the areas of the loops. The same in a restrictive area (a disc).

4. Find transmission and reflection coefficients for a particle, passing over a one-dimensional parabolic barrier *E.Kemble, 1935*

5. Find rate of the atom ionezation by a.c.electric field (the atom might be modeled by a potential well located at the origin) *L.Keldysh, 1964*

6. Find rate of thermo-activated tunneling under Coulomb Barrier (Bethe law) *H.A.Bethe, 1938.*

7. Find cross-section of a neutron scattering on a nucleus, coupled to an environment. *E.Fermi, 1935* Nucleus has mass M and its coupling to environment could be described by the potential $U(\mathbf{r}) = M\ \omega\ \mathbf{r}^2/2$. Temperature of the environment is T

8. Find energy levels of two-dimensional electron in an oscillatory potential and perpendicular magnetic field. *V.A. Fock, 1928; C.G. Darwin, 1931*

9. Find the Green's function of electron on a cyrcle of radius R through which magnetic flux Φ is passing. Determine the energy spectrum.

10. Calculate asymptotics of large n in perturbative expansion of the ground state energy E_0 for anharmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\,\omega^2\,x^2}{2} + \lambda\,\frac{x^4}{4}$$

in powers of λ^n.

11.

12.

13.

14.

15. Calculate the Green's function of a quantum particle in a one-dimensional quatum box. Find its energy levels and the eigenfunctions.

16.

17.

18.

19.

Problem 1

Calculate rate of pair production in external electric field *J.Schwinger, 1948*

S O L U T I O N

Problem 2

Find conditions of quantization for Bloch electron with an arbitrary dispersion law in external magnetic field
L.Onsager, I.Lifshits, 1951

S O L U T I O N

Problem 3

Find distribution function of the winding numbers around the initial point of a random walk in two-dimension. The same for distribution function o the areas of the loops. The same in a restrictive area (a disc).

S O L U T I O N

Problem 4

Find transmission and reflection coefficients for a particle, passing over a one-dimensional parabolic barrier
E.Kemble, 1935

S O L U T I O N

Problem 5

Find rate of the atom ionezation by a.c.electric field (the atom might be modeled by a potential well located at the origin) *L.Keldysh, 1964*

S O L U T I O N

Problem 6

Find rate of thermo-activated tunneling under Coulomb Barrier (Bethe law *H.A.Bethe, 1938*)

S O L U T I O N

Problem 7

Find cross-section of a neutron scattering on a nucleus, coupled to an environment. *E.Fermi, 1935* **Nucleus has mass M and its coupling to environment could be described by the potential $U(\mathbf{r}) = M\omega\mathbf{r}^2/2$. Temperature of the environment is T**

S O L U T I O N

Problem 8

Find energy levels of two-dimensional electron in an oscillatory potential and perpendicular magnetic field.
V.A. Fock, 1928;   C.G. Darwin, 1931

S O L U T I O N

Problem 9

Find the Green's function of electron on a cyrcle of radius R through which magnetic flux Φ is passing. Determine the energy spectrum.

S O L U T I O N

The Green's function obeys Schrödinger equation:

$$\left\{ i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2mR^2}\left(-i\,\frac{\partial}{\partial \phi} - \frac{\Phi}{\Phi_0}\right)^2 \right\} G(\phi, \phi', t) =$$

$$= \delta(t) \sum_{n=-\infty}^{+\infty} \delta(\phi - \phi' + 2\pi n), \quad \Phi_0 = \frac{2\pi\hbar c}{|e|}, \tag{1}$$

and the Green's function $G(\phi, \phi', t)$ is periodic in both angular variables. The solution of Eq (1) with the mentioned condition of periodicity can be written the following form :

$$G(x, x', t) = \sum_{n=-\infty}^{+\infty} \mathcal{G}_t(\delta\phi + 2\pi n), \quad \delta\phi = \phi - \phi' \tag{2}$$

$$\mathcal{G}_t(\phi) = \ \theta(t)\,\sqrt{\frac{imR^2}{2\pi\hbar t}}\ \exp\left[\ \frac{imR^2}{2\hbar t}\ \phi^2 - i\frac{\Phi}{\Phi_0}\ \phi\ \right] \tag{3}$$

Using the Poisson summation formula

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y) \exp\left(2\pi i k y\right) dy, \tag{4}$$

Eq (2) can be rewritten in the form

$$G(\phi, \phi', t) = \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy\ e^{2\pi i l y}\ \mathcal{G}_t(\phi - \phi' + 2\pi y) \tag{5}$$

Calculating the Gaussian integral in y, we arrive at the final expression for the Green's function

$$G(\phi, \phi', t) = \frac{1}{2\pi} \sum_{l=-\infty}^{+\infty} e^{il(\phi-\phi')} \exp\left[-\frac{iE_l(\Phi/\Phi_0)\, t}{\hbar}\right], \tag{6}$$

where the energy $E_l(\theta)$,

$$E_l(\theta) = \frac{\hbar^2}{2mR^2}\, (l-\theta)^2 \tag{7}$$

is shown in Fig 9.1. One can see from this figure the periodicity of the energies in θ with period one, or, therefore, in magnetic flux Φ with period Φ_0. The wave functions $\psi_l(\phi)$ aree equal to :

$$\psi_l(\phi) = \sqrt{\frac{1}{2\pi}}\, e^{il\phi} \tag{8}$$

as it should be on the circle.

Problem 10

Calculate asymptotics of large n in perturbative expansion of the ground state energy E_0 for anharmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\ \omega^2\ x^2}{2} + \lambda\ \frac{x^4}{4}$$

in powers of λ^n.

S O L U T I O N

Problem 15

Calculate the Green's function of a quantum particle in a one-dimensional quantum box. Find its energy levels and the eigen-functions.

S O L U T I O N

The Green's function $G(x, x', t)$ obeys the Schrödinger equation

$$\left\{ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right\} G(x, x', t) = \delta(t)\delta(x - x'), \; 0 \leq x, x' \leq a \tag{9}$$

which obeys the boundary condition:

$$G(0, x') = G(a, x') = 0 \tag{10}$$

Using the mirror image method, the solution of Eq (9) with the boundary condition (10) can be written the form of the following sum[1]

$$G(x, x', t) = \sum_{n=-\infty}^{+\infty} [\mathcal{G}_t(x - x' + 2na) - \mathcal{G}_t(x + x' + 2na)] \tag{11}$$

$$\mathcal{G}_t(x) = \theta(t) \sqrt{\frac{im}{2\pi\hbar t}} \exp\left[\frac{im}{2\hbar t} \, x^2 \right] \tag{12}$$

Using the Poisson summation formula

$$\sum_{n=-\infty}^{+\infty} f(n) = \int_{-\infty}^{+\infty} f(y) dy + 2 \sum_{k=1}^{+\infty} \int_{-\infty}^{+\infty} f(y) \cos(2\pi k y) dy, \tag{13}$$

[1] This sum corresponds to strait-line trajectories which undergo reflections in the walls of the box. After such a line is straighten, it ends at "x" and begins at $x_n^{(\pm)} = \pm x' + 2an$

Eq (11) can be rewritten in the form[2]

$$G(x,x',t) = \\ 2\sum_{k=1}^{+\infty}\int_{-\infty}^{+\infty} dy\cos(2\pi ky)\left[\mathcal{G}_t(x-x'+2ya)-\mathcal{G}_t(x+x'+2ya)\right] \quad (14)$$

Calculating the Gaussian integral in y and using trigonometric identity

$$\cos\left(\frac{x-x'}{2}\right)-\cos\left(\frac{x+x'}{2}\right)=2\ \sin x\ \sin x', \qquad (15)$$

arrive at the final expression for the Green's function

$$G(x,x',t)=\frac{2}{a}\sum_{k=1}^{\infty}\ \sin\frac{\pi xk}{a}\ \sin\frac{\pi x'k}{a}\ \exp\left[-\frac{i\hbar\pi^2k^2}{2ma^2}\ t\right]. \quad (16)$$

This means that the energies and eigen-functions of the particle in the box are equal to:

$$E_k=\frac{\hbar^2\pi^2k^2}{2ma^2}, \qquad \psi_k(x)=\sqrt{\frac{2}{a}}\ \sin\frac{\pi xk}{a} \qquad (17)$$

in complete agreement with very familiar solution of this problem by elementary means.

[2]The simple integral term with $k=0$ in Eq (13) vanishes

References

[1] R.P.Feynmann and A.R.Hibbs *Quantum Mechanics and Path Integrals.* Academic Press

[2] A.Das *Field Theory* World Scientific

[3] A.Das *Thermal Field Theory* World Scientific

[4] L.S.Schulmann *Technique and Applications of Path Integrals* John Wiley

[5] V.N.Popov *Functional Integral and Collective Excitations* Cambridge University Press

[6] A.A.Swidzinsky *Non-uniform problems in theory of superconductvity*

[7] S.Coleman *Aspects on Symmetry* Cambridge University Press

[8] C.Grosche and F.Steiner *Handbook of Feynmann Path Integrals* Springer 1998.

[9] H. Kleinert, *Path Integrals* World Scientific

References

[1] V.G.Vaks and A.I.Larkin *JETP*

[2] A.I.Larkin and S.A.Pikin*JETP*

[3] A.I.Larkin and D.E.Khmelnitskii *JETP*

[4] S.Coleman and E.Weinberg *Phys.Rev*

[5] E.Witten *Nicl.Phys.*

[6] E.Brezin, D.Gross and C.Itzacson *Nucl.Phys*

[7] S.F.Edwards and P.W. Anderson*J.Phys.F*

[8] K.B.Efetov *Adv.Phys.*; *JETP*

[9] L.P.Gorkov *JETP*

[10] A.A.Abrikosov and L.P.Gorkov *JETP*

[11] B.L.Altshuler, A.G.Aronov and D.E.Khmelnitskii *J.Phys*

[12] J.S.Langer and J.Zittartz *Phys.Rev.*

[13] G.E.Volovik *JETP Letters*

[14] J.Klauder *Phys.Rev.*

[15] R.P.Feynmann *Rev.Mod.Phys.***20**, 367 (1948); reprinted in *Quantum Electrodynamics*, ed by J.Schwinger, Dover.