Lecture 7

Boundary problem for kinetic equation. Normal and anomalous skin-effect.

1.Normal Skin-Effect

Everybody knows that metals reflect light like a mirror. Quite clearly, this comes from their good conductance. Indeed, the Maxwell Equations in a metal look as follows

$$\mathrm{curl}\mathbf{H} = \frac{4\pi\sigma\mathbf{E}}{c} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \qquad \mathrm{div}\mathbf{E} = 0 \tag{1}$$

$$\mathrm{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \mathrm{div}\mathbf{H} = 0. \tag{2}$$

where the Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ is used. In these equations, we can neglect displacement current if $4\pi\sigma \gg \omega$. If the Drude formula for conductivity

$$\sigma = \frac{ne^2\tau}{m} \tag{3}$$

is taken into account, then the last criterion can be rewritten as

$$\Omega_L^2 \gg \frac{\omega}{\tau}$$

and derive the resulting equation

$$\nabla^2 \mathbf{H} = -\frac{4\pi i \omega \sigma}{c^2} \mathbf{H} \tag{4}$$

which gives exponential decay of the field strength with the penetration depth δ

$$H(z) = H(0) \exp\{\kappa\ z\}$$

with

$$\kappa = \frac{e^{-\pi/4}}{\delta}, \quad \delta = \sqrt{\frac{c^2}{2\pi\sigma\omega}} \tag{5}$$

High frequency losses under condition of skin effect comes through the product $Q = \mathbf{JE}$, where $\mathbf{J}$ is total surface current which can be expressed through strength of magnetic field $\mathbf{H}(0)$ on the surface

$$J = \int_{-\infty}^{0} j(z) dz = \frac{c}{4\pi} \int_{-\infty}^{0} \frac{dH}{dz} = \frac{c}{4\pi} H(0)$$

Result is that

$$\mathbf{E} = Z(\omega) \mathbf{J}$$

where $Z(\omega)$ is, so called, surface impedance, which is equal to

$$Z(\omega) = R - iX = \sqrt{\frac{2\pi\omega}{\sigma c^2}} e^{-i\pi/4} \tag{6}$$

2. Anomalous Skin Effect and Concept of Non-effectiveness

Anomalous skin effect occurs when $\delta \ll l$ (or $\kappa l \gg 1$). Under these conditions, not all electrons are contributing to the screening but only a small part of them and the effective conductivity is reduced by the factor $1/\kappa l$:

$$\sigma_{eff} \sim \frac{\sigma}{\kappa l} = \frac{ne^2}{mv}\frac{1}{\kappa} \tag{7}$$

where the Eq (3) is used. One can see from Eq. (7) that σ_{eff} does not depend on the mean free path l at all. If to substitute σ_{eff} from Eq. (7) into Eq (5) then a self-consistent condition appears, which has a solution

$$\kappa = \left[\frac{4\pi ne^2}{c^2 p_F}\right]^{1/3} e^{-i\pi/6}, \qquad \delta = \left[\frac{c^2 p_F}{4\pi ne^2}\right]^{1/3} \tag{8}$$

Surface impedance Z under conditions of anomalous skin effect is equal to

$$Z = \left(\frac{4p_F}{ne^2}\right)^{1/3} \left(\frac{\pi\omega}{c^2}\right)^{2/3} e^{-i\pi/3} \tag{9}$$

Condition $\delta \ll l$ for anomalous character of skin effect can be rewritten through frequency ω as

$$\omega \gg \frac{c^2 p_F}{2\pi ne^2 l^3} \tag{10}$$

3.Quantitative Theory

We will consider electrons in the conducting semi-space at $z < 0$, while alternating electric field $\mathbf{E}$ is directed along

x-axis. Dynamics of electrons is govern by the linearised Boltzmann equation:

$$-i\omega_* \delta f + v_z \frac{d\delta f}{dz} + eE(z) v_x \frac{df^{(0)}}{d\epsilon} = 0, \qquad \omega_* = \omega + \frac{i}{\tau} \quad (11)$$

Its solution has the form

$$\delta f(z, \mathbf{v}) = -e \frac{v_x}{v_z} \frac{df^{(0)}}{d\epsilon} \int^z E(z') \exp\left[i \frac{\omega_*}{v_z} (z - z') \right] dz', \quad (12)$$

where direction of electron's velocity is convenient to characterise by its spherical angles θ and ϕ:

$$v_z = v \cos\theta, \qquad v_x = v \sin\theta \cos\phi. \quad (13)$$

The lower limit of integration in Eq (12) should be found separately for electrons, running either to the surface ($\cos\theta > 0$) or from it ($\cos\theta < 0$). Electrons with $\cos\theta > 0$ are coming from the depth of metal, where they have equilibrium distribution. Therefore,

$$\delta f_>(z) = -\frac{ev_x}{v_z} \frac{df^{(0)}}{d\epsilon} \int_{-\infty}^{z} E(z') \exp\left[i \frac{\omega_*(z - z')}{v \cos\theta} \right] dz'. \quad (14)$$

Electrons with $\cos\theta < 0$ are coming after a collision with the surface. Therefore,

$$\begin{aligned} \delta f_<(z) = \delta f_<(0) \; \exp\left[i \frac{\omega_* z}{v \cos\theta} \right] + \\ \frac{ev_x}{v_z} \frac{df^{(0)}}{d\epsilon} \int_z^0 E(z') \exp\left[i \frac{\omega_*(z - z')}{v \cos\theta} \right] dz'. \end{aligned} \quad (15)$$

Distribution function of scattered electrons $\delta f_<(0)$ is connected with that of the incident electrons $\delta f_>(0)$ through the Fuchs condition:

$$\delta f_<(0, -\cos\theta) = \rho \delta f_>(0, \cos\theta), \quad (16)$$

where ρ is coefficient of specular reflection. For completely specular reflection $\rho = 1$, and distribution of reflected electrons coincides with that of the incident once with substitution $\cos\theta \to -\cos\theta$. For completely diffusive reflection $\rho = 0$ and there are no non-equilibrium scattered electrons $\delta f_< \equiv 0$. Therefore, for diffusive surface scattering

$$\delta f_<(z) = \frac{ev_x}{v_z}\frac{df^{(0)}}{d\epsilon}\int_z^0 E(z')\exp\left[i\frac{\omega_*(z-z')}{v\cos\theta}\right]dz', \qquad (17)$$

while for specular surface scattering

$$\delta f_<(z) = \frac{ev_x}{v_z}\frac{df^{(0)}}{d\epsilon}\ \exp\left[i\frac{\omega_* z}{v\cos\theta}\right]\ \int_{-\infty}^0 E(z')\exp\left[-i\frac{\omega_* z'}{v|\cos\theta|}\right]dz$$
$$+\frac{ev_x}{v_z}\frac{df^{(0)}}{d\epsilon}\int_z^0 E(z')\exp\left[i\frac{\omega_*(z-z')}{v\cos\theta}\right]dz' \qquad (18)$$

In the further presentation we will consider only specular scattering.

Current $j(z)$ may be found through non-equilibrium part of distribution function δf:

$$j(z) = \frac{e\nu v}{4}\int d\epsilon\left(\int_0^1 xdx\delta f_>(x,\epsilon,z) + \int_{-1}^0 xdxf_<(x,\epsilon,z)\right) =$$
$$= \int_{-\infty}^{+\infty} dz' K(|z-z'|)E(z'); \qquad (19)$$

where a symmetric continuation of electric field $E(-z) = E(z)$ on the positive semi-axis is assumed. The kernel $K(z)$ in Eq (19) is easier presented by its Furrier transform (remember that $kl \gg 1$)

$$K(z) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\mathcal{K}(k)e^{ikz}, \qquad l_* = v\tau_* = v\left(\frac{1}{\tau} - i\omega\right)^{-1}$$

$$\mathcal{K}(k) = \frac{ne^2}{mv}\int_0^1 \frac{dx}{x}\cdot\frac{xl_*}{1+(xkl_*)^2} = \frac{ne^2}{p_F}\cdot\frac{\pi}{|k|}. \tag{20}$$

Therefore, the wave equation for electric field $E(z)$ has the form:

$$\frac{d^2E}{dz^2} - 2E'(0)\delta(z) = -\frac{4\pi i\omega}{c^2}\int_{-\infty}^{+\infty} dz' K(|z-z'|)E(z'), \tag{21}$$

where the term with δ-function arose due to derivative discontinuity, caused by continuation to the positive semi-axis of z. Eq (21) could be solve by using the Furrier transformation

$$E(z) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\mathcal{E}(k)e^{ikz},$$

$$-k^2\mathcal{E}(k) - 2E'(0) = \frac{ib}{|k|}\mathcal{E}(k), \quad b = \frac{ne^2\omega}{p_F c^2}. \tag{22}$$

As a result,

$$\mathcal{E}(k) = -2E'(0)\frac{|k|}{k^2|k| - ib}, \tag{23}$$

$$E(z) = -\frac{2E'(0)}{\pi}\int_0^\infty \frac{k\cos kz}{k^3 - ib} \tag{24}$$

$$\frac{E(0)}{E'(0)} = -\frac{4}{3\sqrt{3}}\frac{\exp[i\pi/6]}{b^{1/3}}. \tag{25}$$

Left hand side of Eq (25) is directly connected with surface impedance

$$Z = \frac{4\pi i\omega}{c^2}\frac{E(0)}{E'(0)} = \frac{8}{9}\left(\frac{\sqrt{3}\pi p_F\omega^2}{c^4 ne^2}\right)^{1/3}(1 - i\sqrt{3}), \tag{26}$$

which exhibits not only all dependences, obtained by the Pippard's qualitative analysis, but also the relevant coefficients.