Lecture 5.

Diffusion approximation for the Boltzmann equation. Fokker-Planck equation for heavy particle in a gas of light particles. Hot electrons in semiconductors and weakly ionised plasma. Electron temperature, current-voltage characteristics, the energy relaxation rate.

1. Consider a single heavy particle of mass M in the gas of light particles of masses m ($m \ll M$) at temperature T. Mean energy of the light particle $\langle \epsilon \rangle$ is of the order of temperature and its momentum q is of the order of

$$\langle |\mathbf{q}| \rangle \sim \sqrt{mT}, \tag{1}$$

what is significantly smaller than the mean momentum $|\mathbf{p}|$

of the heavy particle

$$\langle|\mathbf{p}|\rangle \sim \sqrt{MT}. \tag{2}$$

Elementary collision processes, therefore, consist of slow variation of momentum of the heavy particle. Therefore, the Boltzmann equation for its distribution function $f(\mathbf{p}, t)$

$$\frac{\partial f}{\partial t} - \mathbf{F}\frac{\partial f}{\partial \mathbf{p}} = \int (d\mathbf{q})\{w(\mathbf{p}+\mathbf{q}, \mathbf{q})f(\mathbf{p}+\mathbf{q}) - w(\mathbf{p}, \mathbf{q})f(\mathbf{p})\} \tag{3}$$

can be simplified, using the Taylor expansion:

$$\begin{aligned} w(\mathbf{p}+\mathbf{q}, \mathbf{q})f(\mathbf{p}+\mathbf{q}) &= w(\mathbf{p}, \mathbf{q})f(\mathbf{p}) \\ +q_\alpha \frac{\partial}{\partial p_\alpha}\{w(\mathbf{p}, \mathbf{q})f(\mathbf{p})\} &+ \frac{q_\alpha q_\beta}{2}\frac{\partial^2}{\partial p_\alpha p_\beta}\{w(\mathbf{p}, \mathbf{q})f(\mathbf{p})\} \end{aligned} \tag{4}$$

As the result, the in absence of external force $\mathbf{F}$ the Boltzmann equation (3) can be presented in the form of a conservation law:

$$\frac{\partial f(\mathbf{p}, t)}{\partial t} + \frac{\partial s_\alpha}{\partial p_\alpha} = 0, \tag{5}$$

$$s_\alpha = -\tilde{A}_\alpha f - \frac{\partial}{\partial p_\beta}(B_{\alpha\beta} f) = -A_\alpha f - B_{\alpha\beta}\frac{\partial f}{\partial p_\beta}; \tag{6}$$

$$\tilde{A}_\alpha = \int q_\alpha w(\mathbf{q})(d\mathbf{q}); \quad B_{\alpha\beta} = \frac{1}{2}\int q_\alpha q_\beta \, w(\mathbf{q})(d\mathbf{q}). \tag{7}$$

Since the equilibrium distribution function

$$f^{(0)}(\mathbf{p}) \propto \exp\left[-\frac{\mathbf{p}^2}{2MT}\right] \tag{8}$$

should vanish the whole collision integral,

$$A_\alpha = B_{\alpha\beta}\,\frac{p_\beta}{MT}. \tag{9}$$

In isotropic system $B_{\alpha\beta} = B\delta_{\alpha\beta}$ and, therefore,

$$\mathbf{s} = -B\left(\frac{\mathbf{p}}{MT}\, f + \frac{\partial f}{\partial \mathbf{p}}\right) \tag{10}$$

Example

Using Eq(3) with collision integral in the form (10), find mobility of a heavy particle in a gas of light particles.

2. We consider now the case of an electron in a weakly ionised plasma, so electron meets only neutral atoms and scatters on them. Since the mass of electron m is much smaller than that of the atom ($m \ll M$), these collisions have a quasi-elastic character, i.e. momentum relaxes efficiently at any collision, while the electron energy changes gradually, obeying a diffusion equation. The resulting kinetic equation keeps the form:

$$\frac{\partial f}{\partial t} - e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}} = -\frac{1}{p^2}\frac{\partial(p^2 s_p)}{\partial p} + Nv\int\{f(\theta') - f(\theta)\}d\sigma, \tag{11}$$

where

$$s_p = -B\left(\frac{p}{mT}f + \frac{\partial f}{\partial p}\right). \tag{12}$$

2.2 In order to evaluate the energy transfer in the elementary collision act of an electron with velocity $\mathbf{v}$ with an atom with velocity $\mathbf{V}$ (velocities after the collision are $\mathbf{v}'$ and $\mathbf{V}'$ respectively), let us note that the energy conservation in the centre of mass frame leads to following identity

$$(\mathbf{v} - \mathbf{V})^2 = (\mathbf{v}' - \mathbf{V}')^2, \tag{13}$$

which could be transform to even simpler form

$$2(\mathbf{v}-\mathbf{v}')\mathbf{V} = v^2 - v'^2 \approx 2v\Delta v, \tag{14}$$

taking into account that $\mathbf{V} \simeq \mathbf{V}'$. From Eq (14) follows the expression for the momentum transfer Δp:

$$(\Delta p)^2 = \frac{m^2}{v^2}\left[(\mathbf{vV})^2 + (\mathbf{v}'\mathbf{V})^2 - 2(\mathbf{vV})(\mathbf{v}'\mathbf{V})\right]. \tag{15}$$

Since thermal average gives $\langle V_\alpha V_\beta\rangle = \langle V^2\rangle\delta_{\alpha\beta}/3 = T\delta_{\alpha\beta}/M$,

$$(\Delta p)^2 = \frac{2m^2T}{M}(1-\cos\alpha), \tag{16}$$

where α is the scattering angle. One can see from that the Diffusion coefficient B in modulus of momentum is equal to

$$B = \frac{Nm^2vT\sigma_{\mathrm{tr}}}{M} = \frac{mTp}{Ml_{\mathrm{tr}}}; \quad l_{\mathrm{tr}} = \frac{1}{N\sigma_{\mathrm{tr}}}, \tag{17}$$

and the flux s_p is given by the following expression

$$s_p = -\frac{mp}{Ml_{\mathrm{tr}}}\left(\frac{p}{m}f + T\,\frac{\partial f}{\partial p}\right). \tag{18}$$

2.3. We can return now to Eq (11) assuming that nothing depends on time t. The term with electric field $\mathbf{E}$ in it could be transformed into the following form

$$e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}} = eE\frac{\partial f}{\partial p_z} = eE\left[\cos\theta\frac{\partial f}{\partial p} + \frac{\sin^2\theta}{p}\frac{\partial f}{\partial\cos\theta}\right]. \tag{19}$$

Since the elastic scattering is strong, we can keep only first two terms f_0 and f_1 in the expansion of $f(p,\cos\theta)$ in Legendre polynomials

$$f(p,\cos\theta) = \sum_{n=0}^{\infty} f_n(p)P_n(\cos\theta) \simeq f_0(p) + f_1(p)\cos\theta, \tag{20}$$

and $f_0 \gg f_1$.
The elastic collision term in Eq (11) can be rewritten in the form

$$N v \int \{f(p,\theta') - f(p,\theta)\} d\sigma = -\frac{v}{l_{\rm tr}} f_1 \cos\theta. \qquad (21)$$

Neglecting diffusion term with s_1 compare with s_0, we can, finally, rewrite Eq (11) in the form

$$-eE\{f_0' \cos\theta + f_1' \cos^2\theta + \frac{f_1}{p}\sin^2\theta\}$$
$$+\frac{1}{p^2}\frac{d}{dp}\left(p^2 s_0\right) + \frac{v}{l_{\rm tr}} f_1 \cos\theta = 0. \qquad (22)$$

Projecting the left hand side of Eq (22) to f_0 and f_1, we obtain two equations

$$\frac{1}{p^2}\frac{d}{dp}\left(p^2 s_0\right) - \frac{2}{3}\frac{eEf_1}{p} - \frac{1}{3}eEf_1' = 0, \qquad (23)$$

$$f_1 = \frac{eEl_{\rm tr}}{v} f_0. \qquad (24)$$

Eq (23) can be re-written in the form

$$\frac{1}{p^2}\frac{d}{dp}\left(p^2 S_0\right) = 0; \quad S_0 = s_0 - \frac{1}{3}eEf_1. \qquad (25)$$

Solution of Eqs (23, 25) has, therefore, the form $S_0 = C/p^2$, while the boundary condition at $p \to \infty$ fixes the constant $C = 0$ and, finally, using Eq (24), we obtain the equation

$$S_0 = 0 = -\frac{p^2}{Ml_{\rm tr}} f_0 + \left(-\frac{mTp}{Ml_{\rm tr}} - \frac{(eE)^2 m\, l_{\rm tr}}{3p}\right) f_0'. \qquad (26)$$

Since the function f_0 depends only on modulus p of momentum, it is convenient to re-write Eq (26), using the energy $\epsilon = p^2/2$ as is argument:

$$f_0 + \left(T + \frac{(eEl_{\rm tr})^2}{6\epsilon}\,\frac{M}{m}\right)\frac{df_0}{d\epsilon} = 0. \tag{27}$$

Equation (27) has the following solution:

$$f_0(\epsilon) = \left(\frac{\epsilon}{T} + \frac{\gamma^2}{6}\right)^{\gamma^2/6} \exp\left[-\frac{\epsilon}{T}\right], \quad \gamma = \frac{eE\,l_{\rm tr}}{T}\sqrt{\frac{M}{m}}. \tag{28}$$

Knowledge of zero harmonic $f_0(\epsilon)$ allows to find the first harmonic f_1:

$$f_1(\epsilon) = -f_0\sqrt{\frac{M}{m}}\frac{\gamma\epsilon}{\epsilon + \gamma^2 T/6}, \quad \gamma = \frac{eE\,l_{\rm tr}}{T}\sqrt{\frac{M}{m}}. \tag{29}$$

Asymptotes of this solution give either linear regime at $\gamma \ll 1$ or strongly nonlinear regime at $\gamma \gg 1$. At $\gamma \ll 1$ distribution function f_0 coincides with the Maxwellian one $\exp[-\epsilon/T]$, while

$$f_1 = \frac{eE\,l_{\rm tr}}{T} f_0. \tag{30}$$

Electric field leads to drift of electrons with the averaged velocity $\mathbf{v} = eb_0\mathbf{E}$, where mobility b_0 could be obtained using Eq (30)

$$b_0 = \frac{2^{2/3}\,l_{\rm tr}}{3\pi^{1/2}}\sqrt{mT}. \tag{31}$$

At $\gamma \gg 1$ the distribution function f_0 has a Maxwellian tail, while in the significant range of moderate energies it has the form[1]

$$f_0 \simeq \exp\left[-\frac{3\epsilon^2}{\gamma^2 T^2}\right]. \tag{32}$$

The mean value of energy $\langle\epsilon\rangle$ of electrons

$$\langle\epsilon\rangle \simeq \gamma T = eE\, l_{\rm tr}\sqrt{\frac{M}{m}} \tag{33}$$

strongly exceeds, under this condition, the temperature of neutral atoms T, which forms for electron a thermal bath. This is why this situation is often called "the hot electrons". In the regime of electrons, "heated" by the electric field, the current I depends on field in a non-linear fashion:

$$I \simeq (mM)^{1/4} l_{\rm tr}^{1/2} (eE)^{3/2}. \tag{34}$$

2.3. The results for the "hot electrons" regime could be understood better, if we return to the time dependent equation (11), keep only two angular harmonics of the distribution function and use Eq (24). If we multiply the resulting equation we by $\epsilon = p^2/2m$ and integrate it over p, we obtain the equation of the energy balance

$$\frac{d}{dt}\langle\epsilon\rangle = -\frac{\langle\epsilon\rangle - 3T/2}{\tau_{\rm in}} + \frac{e^2E^2\, l_{\rm tr}}{6}\sqrt{\frac{1}{m\langle\epsilon\rangle}}, \tag{35}$$

where

$$\frac{1}{\tau_{\rm in}} = \frac{1}{l_{\rm tr}}\sqrt{\frac{\langle\epsilon\rangle}{M}}. \tag{36}$$

[1] In order to obtain this asymptote, re-write the square bracket in Eq (28) in the exponential form and expand the logarithm in inverse powers of the large parameter γ^2

Eq (35) expresses the energy balance through the rates of energy relaxation τ_{in} and heating by the electric field. Under stationary conditions, both right and left-hand sides of Eq (35) vanishes. If the electric field is weak, than the heating could be neglected and $\langle\epsilon\rangle = 3T/2$. In the opposite case, the action of the bath could be neglected and $\langle\epsilon\rangle \simeq eE\, l_{\mathrm{tr}}\sqrt{M/m}$. The mobility b_0 in the linear regime is proportional to the thermal velocity $\sqrt{mT}$, as well as in the non-linear regime, while $b \simeq \sqrt{m\langle\epsilon\rangle} \sim \sqrt{eE}(M/m)^{1/4}$. This last estimate explains the result of Eq (34).
Finally, we can see that the expression for the mobility (31) in the linear regime consists of the elastic mean free path only, while the criterion of linearity consists of the rate of inelastic collisions (36).