Lecture 2

Diffusion equation, linear response, conductivity and the Einstein relations.τ - approximation for collision integral. Magneto-resistance, the Hall effect and thermo-power for electrons in metals. Hydrodynamics derived from kinetics.

We can consider now collisions with static impurities as a dominant scattering process. The Boltzmann equation under this conditions looks like the following:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}} = I_{\text{imp}}\{f\} + \text{St}\{f\}; \quad (1)$$

$$I_{\text{imp}}\{f\} = \sum_{\mathbf{p}_1} w(\mathbf{p}, \mathbf{p}_1)[f_{\mathbf{p}_1}(1 - f_{\mathbf{p}}) - f_{\mathbf{p}}(1 - f_{\mathbf{p}_1})] =$$

$$= \sum_{\mathbf{p}_1} w(\mathbf{p}, \mathbf{p}_1)(f_{\mathbf{p}_1} - f_{\mathbf{p}}), \quad (2)$$

where $\mathrm{St}\{f\}$ stays for the collision integral of inelastic collisions.

Integrating Eq (1) over momentums $\mathbf{p}$, and taking into account that the collision integral $\mathrm{St}\{f\}$ obeys the particle conservation law $\int (d\mathbf{p}) \mathrm{St}\{f\} = 0$, we obtain the continuity condition:

$$\frac{\partial n}{\partial t} + \frac{1}{e} \mathrm{div} \mathbf{j} = 0. \quad (3)$$

Electric current $\mathbf{j}$ arise due to two reasons: electric field $\mathbf{E}$ and inhomogeneity of electron density $n(\mathbf{r})$. In the linear regime

$$\mathbf{j} = \sigma \mathbf{E} - eD\nabla n, \quad (4)$$

where σ is conductivity and D the diffusion coefficient. This two kinetic coefficients are not independent. To find their relations, consider an electrically isolated piece of conductor in a steady electric filed $\mathbf{E} = \nabla\phi(\mathbf{r})$ (ϕ is electrostatic potential). The electron distribution function is at its stationary value $f^0(\mathbf{p}, \mathbf{r})$. Inelastic collision integral $\mathrm{St}\{f^{(0)}\}$ at this function is zero. As for for the elastic collision term $I_{\mathrm{imp}}\{f\}$, it is zero if $f^0(\mathbf{p}, \mathbf{r}) = f^0[\epsilon(\mathbf{p}), \mathbf{r}]$. This all means that the only space dependence in the distribution function comes through the chemical potential $\mu(\mathbf{r})$:

$$f^0(\mathbf{p}, \mathbf{r}) = f^0[\epsilon(\mathbf{p}) - \mu(\mathbf{r})] = \left(\exp\left[\frac{\epsilon(\mathbf{p}) - \mu(\mathbf{r})}{T}\right] \pm 1 \right)^{-1} \quad (5)$$

Kinetic equation in this case has the form

$$\frac{df^{(0)}}{d\epsilon} (-\mathbf{v}\nabla\mu - e\mathbf{v}\nabla\phi) = 0, \quad (6)$$

which leads to condition of equality of the electro-chemical potential

$$\mu - e\phi = \mu_0 = \text{const.} \tag{7}$$

Current is zero in this isolated sample $\mathbf{j} = 0$, which leads to

$$\sigma \nabla \phi = eD \frac{\partial n}{\partial \mu} \cdot \nabla \mu; \quad \sigma = e^2 D \frac{\partial n}{\partial \mu}. \tag{8}$$

Eq (8) exhibits, the so called, Einstein relation. Substituting Eqs (4) and (8) into Eq (3), and taking into account the electrostatic equation

$$\text{div}\mathbf{E} = 4\pi e n \tag{9}$$

we find the equation, that govern the relaxation of an extra charge to its equilibrium distribution (Maxwell relaxation)

$$\frac{\partial n}{\partial t} - D\nabla^2 n + 4\pi n D e^2 \frac{\partial n}{\partial \mu} = 0. \tag{10}$$

If some leads are attached to a conductor and a voltage is applied to it, then electric current $\mathbf{j}$ passes through even when the density is uniform. In the leading order in electric field $\mathbf{E}$ the distribution function looks like

$$f(\mathbf{p}) = f^{(0)}(\epsilon_{\mathbf{p}}) + \delta f.$$

The Eq (1) has under this condition the form

$$e\mathbf{v}\mathbf{E}\frac{df^{(0)}}{d\epsilon} = I_{\text{imp}}\{\delta f\} = \sum_{\mathbf{p}_1} w(\mathbf{p}, \mathbf{p}_1)(\delta f_{\mathbf{p}_1} - \delta f_{\mathbf{p}}). \tag{11}$$

Solution of this equation is naturally to search in the form

$$\delta f_{\mathbf{p}} = \mathbf{v}\mathbf{E}\frac{df^{(0)}}{d\epsilon} g(\epsilon_{\mathbf{p}}),$$

which immediately gives

$$\delta f_{\mathbf{p}} = -\mathbf{vE}\frac{df^{(0)}}{d\epsilon}\tau_{\mathrm{tr}} \tag{12}$$

$$\frac{1}{\tau_{\mathrm{tr}}} = \sum_{\mathbf{p}_1} w(\mathbf{p},\mathbf{p}_1)\left(1-\frac{\mathbf{pp}_1}{p^2}\right) = \sum_{\mathbf{p}_1} w(\mathbf{p},\mathbf{p}_1)(1-\cos\theta) \tag{13}$$

Angle θ in Eq (12) is one between directions of $\mathbf{p}$ and $\mathbf{p}_1$. Electric current $\mathbf{j}$ obeys to the Ohm's law:

$$\mathbf{j} = \int (d\mathbf{p}) e\mathbf{v}\delta f(\mathbf{p}) = \sigma\mathbf{E} \tag{14}$$

$$\sigma = \frac{e^2\tau_{\mathrm{tr}}v^2\nu}{3}\int d\epsilon\left(-\frac{df^{(0)}}{d\epsilon}\right) = \frac{ne^2\tau_{\mathrm{tr}}}{m} \tag{15}$$

$$\nu(\epsilon) = \int (d\mathbf{p})\delta(\epsilon-\epsilon_{\mathbf{p}}) = \frac{mp_F}{2\pi^2\hbar^3}, \quad n = \frac{4\pi}{3}\left(\frac{p_F}{2\pi\hbar}\right)^3 \tag{16}$$

ν stays for the density of states and n for electron density. Eq (15) exhibits famous Drude formula for conductivity. The Einstein relation gives an equivalent formula for diffusion coefficient

$$D = \frac{v^2\tau_{\mathrm{tr}}\nu}{3}. \tag{17}$$

Relaxation time τ_{tr} gives an inverse rate of momentum relaxation[1] Eq (11) is a linearized equation of the type, which always arises, when a liner response of certain type is calculated. Since the right hand side is an integral operator and its kernel depends only on the angle θ between

[1] Note that elastic collisions give a finite value of conductance and inelastic collisions are not needed to find it. This could, of course, lead to apparently wrong impression that a dissipative current flow does not need for its stabilization any energy relaxation. We return to this puzzle in these lectures later, when the energy relaxation will be considered in details

directions of $\mathbf{p}$ and $\mathbf{p}_1$, everything depends on angular dependence of the left-hand side of Eq (11). If the left-hand side of Eq (11) contains $P_n(\cos\theta)$, then the collision integral I_{imp} could be reduced to, so called, relaxation rate form:

$$I_{\text{imp}} = -\frac{f-\bar{f}}{\tau_n}, \quad \frac{1}{\tau_n} = \sum_{\mathbf{p}_1} w(\mathbf{p},\mathbf{p}_1)[1-P_n(\theta)]. \tag{18}$$

Here $\bar{f}$ stays for the distribution function, averaged over directions of momentum. In all transport phenomena the momentum relaxation time $\tau_{\text{tr}} = \tau_1$ appears[2].

Consider now transport in presence of magnetic field $\mathbf{B}$ in electric field $\mathbf{E}$, perpendicular to magnetic one ($\mathbf{E}\cdot\mathbf{B} = 0$). The linearized Boltzmann equation takes now the form:

$$e\mathbf{v}\mathbf{E}\frac{df^{(0)}}{d\epsilon} + \frac{e}{c}[\mathbf{v}\mathbf{B}]\frac{\partial\delta f}{\partial\mathbf{p}} = -\frac{\delta f}{\tau_{\text{tr}}}. \tag{19}$$

Introducing angle α between direction of electric field $\mathbf{E}$ and a component of momentum $\mathbf{p}_\perp$, perpendicular to magnetic field, Eq (19) could be rewritten in the form

$$\frac{ep_\perp}{m}E\cos\alpha\frac{df^{(0)}}{d\epsilon} + \Omega\frac{\partial\delta f}{\partial\alpha} = -\frac{\delta f}{\tau_{\text{tr}}}, \quad \Omega = \frac{eB}{mc}. \tag{20}$$

Solution of Eq (20) has the form

$$\delta f = \frac{ep_\perp}{\Omega m}E\left(-\frac{df^{(0)}}{d\epsilon}\right)\text{Re}\int_{-\infty}^{\alpha} d\alpha' \exp\{i\alpha' - \frac{\alpha-\alpha'}{\Omega\tau_{\text{tr}}}\} \tag{21}$$

$$= \frac{ep_\perp}{m}E\left(-\frac{df^{(0)}}{d\epsilon}\right)\frac{\tau_{\text{tr}}}{1+(\Omega\tau_{\text{tr}})^2}(\cos\alpha + \Omega\tau_{\text{tr}}\sin\alpha) \tag{22}$$

[2] Of course, the relaxation rate form (18) is much simpler and more convenient, than an integral collision operator. Note, that reduction to this form is exact for elastic collisions.

Calculating the current, we arrive to the well known expression of Drude theory

$$\mathbf{j} = \sigma_{xx}\mathbf{E} \ + \ \sigma_{xy}\frac{[\mathbf{EB}]}{B}; \tag{23}$$

$$\sigma_{xx} = \frac{ne^2}{m} \cdot \frac{\tau_{\mathrm{tr}}}{1+(\Omega\tau_{\mathrm{tr}})^2}; \qquad \sigma_{xy} = \frac{ne^2}{m} \cdot \frac{\Omega\tau_{\mathrm{tr}}^2}{1+(\Omega\tau_{\mathrm{tr}})^2}. \tag{24}$$