# Lecture 15.
# Negative Magnetoresistance. Weak Localization and Scaling Theory for Anderson Localization

## The Puzzle of Negative Magnetoresistance

This puzzle has been known for about 35 years from late 40-th to 1979 when its resolution was been found. The main unusual features of this phenomenon are:

1. Magnetoresistance is negative, i.e. magnetic field enhances rather than suppresses conductance ;

2. Magnetoresistance is isotropic, i.e. does not depend on the angle between the current $\mathbf{j}$ and magnetic field $\mathbf{H}$;

3. The dependence of resistance on magnetic field saturates for relatively weak fields, well before the Larmor radius $R_L$ becomes comparable with the mean free path $l$;

4. For very weak fields magnetoresistance is proportional to $H^2$, but the proportionality factor is too large and increase at low temperatures. If one uses the Drude formula to describe it, then the characteristic temperature dependent time is several order of magnitude larger, than the mean free time $\tau$, that could be estimated from conductivity;

5. In the very end (1978), it was found that negative magnetoresistance of thin films and inversion layers depends upon

the component of magnetic field $H_{\perp}$ perpendicular to their surface. This, undoubtedly, shows that the whole effect has an orbital origin.

## Weak Localization. Quantum Correction to Conductivity. Effective Dimension

To solve the puzzle of magnetoresistance, let us first analyze the general concept of transport in Quantum Mechanics. At low temperatures, the rate of all inelastic processes is very small and we can neglect it in the leading approximation. Therefore, if we have a source of electrons is at some point $S$ and a detector at a point $D$ in Fig.1, then the electrons propagate along the "rays", emitted from the source and then scattered by impurities on their way to the detector. According to the general rule of Quantum Mechanics, an amplitude $A_i$ should be associated with every trajectory. The total probability $W$ of detecting electron

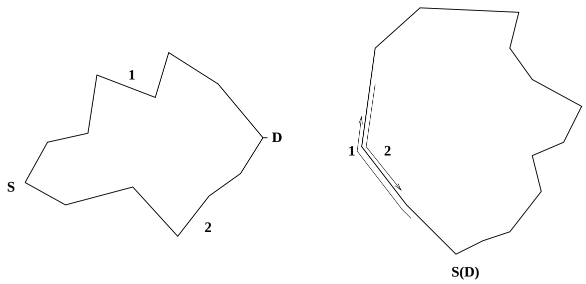


Figure 1: Typical diffusing trajectories from the source 1 to drain 2

is

$$W = |\sum_i A_i|^2 = \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_j^* \tag{1}$$

The first term is a sum of classical probabilities of an electron moving along particular trajectories, while the second one takes care of the interference of partial amplitudes. The classical probabilities can be found from the Boltzmann equation. It leads to the classical diffusion with the coefficient $D = v^2\tau/d$, where $v$ is the velocity of electrons (the Fermi velocity $v_F$ under the condition of degeneracy), $\tau$ is the mean free time and $d$ is the dimension. The effect of interference is usually neglected. The reason for that is that the mean free path $l = v\tau$ is much larger than the wave length of electrons, $\lambda = \hbar/p_F$. Therefore, the second term in Eq (1) is a sum of strongly oscillating terms, which cancel each other, but they are not canceled completely. We may see this from a simple example of the case, when the source $S$ and detector $D$ are located at the same point. Then, every closed trajectory, that begins at $S$ and ends at $D$, can e traversed in two opposite directions: clockwise and counterclockwise. The amplitudes $A_1$ and $A_2$, corresponding to these directions, are coherent and the probability they give is $W = 4|A_1|^2$ instead of $2|A_1|^2$, one would expect from the classical analyses. Therefore, quantum effects contribute to the probability of returning to the initial point, of random walk, effectively diminish the conductivity $\sigma$ and diffusivity $D$. This contribution is large, but comes from those very rare trajectories, which return to the initial point.

In a precise quantum mechanical analysis, instead of trajectories, one has to consider the wave pipes of the radius equal to the wave length $\lambda$. "To return to initial point" means to achieve an intersection of the wave pipes, so, the initial point is located inside the intersecting pipe (see Fig). The classical probability $dW$ of passing at time $t$ through a volume element $dV$ at the distance $\mathbf{r}$ from the origin is equal to

$$dW \sim \frac{dV}{(Dt)^{d/2}} \exp\left[-\frac{r^2}{2Dt}\right]. \tag{2}$$

Finding effect of interference as an integral over the time $t$ of return and using Eq (2) at $\mathbf{r} = 0$, we obtain the following estimate of the relative correction to conductivity:

$$\frac{\delta\sigma}{\sigma} \sim -\int_{\tau}^{\tau_\phi} \frac{\lambda^{d-1} v dt}{(Dt)^{d/2}}. \tag{3}$$

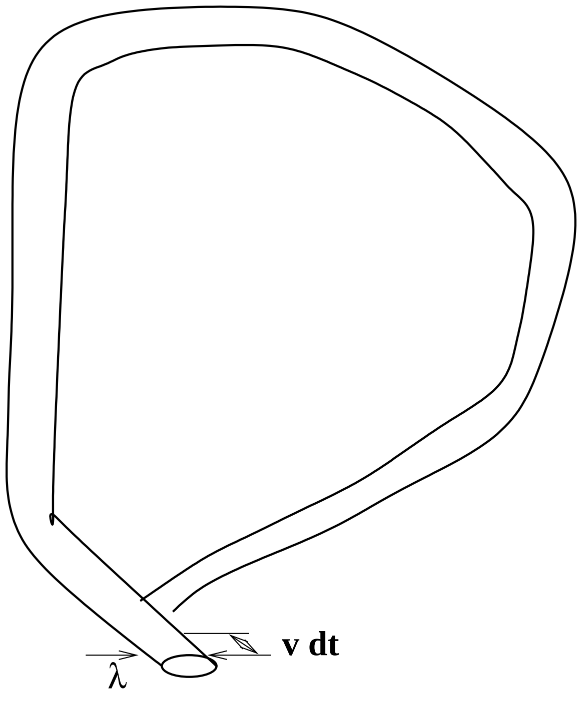


Figure 2: To estimate of the quantum correction to conductivity

The upper limit in this integral corresponds to the characteristic time $\tau_\phi$ of electron phase memory relaxation due to inelastic processes, the lower limit is the mean free time $\tau$, because there are no returns at times $t$, smaller than $\tau$. Taking into account the Einstein relation $\sigma = e^2\nu D$ ($\nu$ is the density of states), we obtain

$$\delta\sigma \sim \frac{e^2}{\hbar}\begin{cases} L_\phi, & d=1, \\ \log(\tau_\phi/\tau), & d=2, \\ \text{const} - L_\phi^{-1}, & d=3, \end{cases} \tag{4}$$

where $L_\phi = \sqrt{D\tau_\phi}$ is the characteristic length of coherent diffusive propagation.

Formulae (4) for the quantum correction to conductivity exhibit a phenomenon, called "Weak Localization": even in well conducting materials ($\sigma \gg e^2/\hbar$) quantum processes suppress conductivity. For $d = 1, 2$ these corrections are divergent at $\tau_\phi \to \infty$ , which means that all states are localized for an arbitrary weak disorder and, at $T = 0$, the conductance vanishes. For $d = 3$, the quantum correction is finite, so the conductance remains finite at $T = 0$, provided disorder is weak. If disorder is stronger, than a certain value, the electronic states are localized. This means that there is a transition from localized to extended states for $d = 3$ as a function of either Fermi energy or strength of disorder (the Anderson transition).

It is now natural to ask what the dimension $d$ in Eqs (3) and (4) means. To get an answer, let us consider random walk in a film of finite thickness $a \gg l$. At small times $t \ll a^2/D$, a diffusive trajectory returns back to its initial point with probability, proportional to $(Dt)^{-3/2}$. At larger times $t \gg a^2/D$, this probability is proportional to $(Dt)^{-1}$. Therefore, if phase breaking time $\tau_\phi$ is larger, than $a^2/D$, then this film behaves like a two-dimensional conductor. Otherwise, it is a bulk $3D$ conductor. The criteria for dimensional crossover looks like

$$a \qquad \begin{cases} < L_\phi, & d = 2, \\ > L_\phi, & d = 3. \end{cases} . \tag{5}$$

## Magnetoresistance

Now we are ready to analyze the effect of magnetic field on quantum corrections to conductivity. Despite of being small, this effect is so sensitive to magnetic field, that it determines the whole magnetoresistance. The magnetic field $B$ influences quantum mechanics of an electrons in two ways: i). the Lorentz force curves the electron's trajectory; as a result, the direction of the electron's momentum deviates from its initial direction; the deviation of momentum direction $\theta$ in the mean free time $\tau$ is of the order of

$$\Delta\theta \sim \frac{evB\tau}{cp} = \Omega\tau, \quad \Omega = \frac{eB}{mc};$$

ii). the amplitudes of travesting the loop clockwise and counter-clockwise get extra phase factors (the Aharonov-Bohm effect):

$$A_{1,2} \to A_{1,2} \exp\left[\pm i \frac{e}{c\hbar} \int \mathbf{A} d\mathbf{r}\right].$$

As one can see, the mechanism i). is completely classical and is suppressed if the mean free time $\tau$ is short. The mechanism ii). is quantum and is not limited by any elastic processes. This is why the latter gives stronger dependence of the conductance on

the magnetic field.
As a result of Aharonov-Bohm effect, the interference term in Eq (1) gets extra factor, $\cos(eBS/c\hbar)$, where $S$ is the area encircled by the loop. For a loop of length $\mathcal{L} = vt$, this area can be estimated as $S(t) \sim Dt$. So, the estimate for magnetoresistance is

$$\Delta\sigma^{(d)} = \sigma(B) - \sigma(0) \sim$$
$$\sim \frac{e^2}{\hbar} \cdot \int_0^{\tau_\phi} \frac{\lambda^{d-2} dt}{D^{(d-2)/2} t^{d/2}} \left[ 1 - \cos\left( \frac{eBDt}{c\hbar} \right) \right] \tag{6}$$

Introducing a dimensionless variable, we can simplify the expression on the right hand side of Eq (6):

$$\Delta\sigma^{(2)} \sim \frac{e^2}{\hbar} \cdot f_2\left(\frac{eBD\tau_\phi}{c\hbar}\right); \; f_2(x) = \int_0^x \frac{dz}{z}[1 - \cos z]; \tag{7}$$
$$\Delta\sigma^{(3)} \sim \frac{e^2}{\hbar}\sqrt{\frac{eB}{c\hbar}} \cdot f_3\left(\frac{eBD\tau_\phi}{c\hbar}\right); \; f_3(x) = \int_0^x \frac{dz}{z^{3/2}}[1 - \cos z]. \tag{8}$$

The functions $f_{2,3}(x)$ have the following asymptotes:

$$f_2(x) \sim \begin{cases} x^2, & x \ll 1, \\ \log x, & x \gg 1, \end{cases} \tag{9}$$
$$f_3(x) \sim \begin{cases} x^2, & x \ll 1, \\ x^{1/2}, & x \gg 1. \end{cases} \tag{10}$$

These formulae explain the mystery of the negative magnetoresistance: the conductivity consists of the Drude part $\sigma^{(0)}$ and a negative quantum correction $\delta\sigma$ due to weak localization; the Drude part is suppressed by magnetic field through the classical parameter $\Omega\tau$; the quantum correction is suppressed much stronger, which leads to enhancement of the conductivity. The magnetoresistance does not depend on the relative directions of the current and magnetic field, because it is determined by the loops, which can have any orientation, provided the sample is three-dimensional. In $2D$-systems the loops can be located only in the plane of the sample and, therefore, the conductance depends on the perpendicular component of the field only. The saturation in relatively weak fields and the temperature dependent behavior at extremely low fields can also bee attributed to the Aharonov-Bohm effect on the quantum correction. As it is seen from Eqs (7) and (8), the mean free time $\tau$ of elastic collisions does not enter the expressions (7) and (8) for magnetoresistance $\Delta\sigma(H)$. Instead, the magnetoresistance depends on phase breaking time $\tau_\phi$, which is much longer than $\tau$ and depends on temperature.

## The Aharonov-Bohm effect

The most direct way of verification of the coherent mechanism of magnetoresistance is provided by observing the magnetoresistance of hollow cylinders. Consider a cylinder of radius $R$, circumference $L = 2\pi R$ and height $h$, made of a film of thickness $a \ll R$. The magnetic field $\mathbf{B}$ is parallele to cylinder's axis. How does the resistance of this cylinder depends on filed ?
Before answering this question, let us find the magnetoresistance of a flat film in a parallel field. We assume that the thickness of the film is much smaller, than the magnetic length ($a \ll \lambda_H = \sqrt{c\hbar/eB}$) and much larger, than the free path ($a \gg l = v\tau$). The projection of the electron's trajectory incide the film on the plane, perpendicular to the direction of the

magnetic field **B**, has many self-crossings (see Fig). Therefore, a loop of length Ł $= vt$ has $N_t \sim Dt/a^2$ sub-loops. The flux $\Phi$ through any of these sub-loops is of order $\Phi \sim Ba^2$, but but fluxes through different sub-loops have a random sign. As the result, the quantum correction to conductivity obtains an extra factor, $\cos[e\Phi/c\hbar] \approx 1 - (e\Phi/c\hbar)^2/2$, for every sub-loop and the factor

$$\exp[-\frac{t}{\tau_B}], \quad \frac{1}{\tau_B} \sim D\left(\frac{eBa}{c\hbar}\right)^2 = \frac{Da^2}{\lambda_B^4}$$

for the whole loop. Therefore, the phase breaking time is modified in the presence of magnetic field:

$$\frac{1}{\tau_\phi(B)} = \frac{1}{\tau_\phi} + \frac{1}{\tau_B}, \tag{11}$$

and the magnetoresistance of the film in a parallel field is

$$\Delta\sigma(B) \sim \frac{e^2}{\hbar} \cdot \log\left[\frac{\tau_\phi}{\tau_\phi(B)}\right]. \tag{12}$$

Now we are ready to consider the magnetoresistance of a hollow cylinder. It comes from the loops, winding around the cylinder one, two, three, .. $n$ ... times. So,

$$\Delta\sigma(B) = \frac{2e^2}{\hbar}\int_0^{\tau_\phi} dt \sum_{n=0}^{\infty} |A_n(t)|^2 \cos\left[4\pi\left(\frac{\pi BR^2}{\Phi_0}\right)\right], \tag{13}$$

where

$$\Phi_0 = \frac{2\pi\hbar c}{e},$$

and the asymptote of $|A_n|^2$ is

$$|A_n|^2 \sim \exp\left[\frac{(2\pi Rn)^2}{Dt}\right]$$

Equation (13) shows that the conductance oscillates as a function of magnetic flux through the cylinder $\Phi = \pi B R^2$ with the period of $\Phi_0/2$. These oscillations decay at higher temperatures, if the circumference of the cylinder is larger than the diffusion length of the dephasing ($2\pi R \gg L_\phi = \sqrt{D\tau_\phi}$ ). The lower is the temperature, the many harmonics $n$ the oscillations contain. The finite thickness of the film has two effects: the oscillations get a background $A_0^2(B)$; the dephasing rate $\tau_\phi^{-1}$ should be replaced in all formulae by the effective rate $\tau_\phi^{-1}(B)$ from Eq (11). The experimental observation of the AB-oscillations of magnetoresistance by Yu. and D.Sharvin has become a true victory of the whole concept of weak localization.

## Effect of Spin-Orbit Interaction

Recalling that electrons can have intrinsic quantum numbers, like spin projection, the energy band index, etc, one would naturally ask whether these factors can affect weak localization. The answer is:"They could", but let us go through it in the order.

Suppose that an electron can belong to one of two pockets (the index $\alpha$ takes on two values: either 1 or 2) in the momentum space (valleys of the energy spectrum). Then the amplitudes $A_{1,2}$ are matrices in the valley indices: $A_{1,2}^{\alpha\beta}$. The quantum correction to conductivity can be expressed as

$$\int dt C^{\alpha\beta\beta\alpha}(r,r,t) \propto \mathrm{Re} \int dt A_1^{\alpha\beta}(r,-t;r,t)[A_2^{\beta\alpha}(r,-t;r,t)]^*,$$

where $C^{\alpha\beta\gamma\delta}(t)$ is the interference amplitude, the indices of which correspond to time $-t$ (first two) and to time $t$. This matrix must be real and diagonal in these pairs of indices:

$$C^{\alpha\beta\gamma\delta}(t) = C_s \delta^{\alpha\beta}\delta^{\gamma\delta}(t) + C_a \tau_3^{\alpha\beta}\tau_3^{\gamma\delta},$$

where $\tau_3$ is the Pauli matrix and $C_{s,a}$ are either symmetric $(s)$ or anti-symmetric $(a)$( with respect to the valley permutation ) phase correlators. If there is no inter-valley scattering and at time $-t$ only valley 1 is populated, then only $C^{1111}$ is not equal to zero. Therefore, at $t = 0$

$$C_s = C_a = \frac{1}{2}.$$

At $t \neq 0$ the valley 1 remains populated and the valley 2 remains empty. So, both symmetric and antisymmetric correlators decay with the same phase relaxation time $\tau_\phi$:

$$C_{s,a} \propto \frac{1}{2} \exp\left(-\frac{t}{\tau_\phi}\right)$$

If there is an inter-valley relaxation, the asymmetric correlator decays at its own rate $\tau_v^{-1}$ and

$$C^{\alpha\beta\beta\alpha} = \frac{1}{2}\left[\exp\left(-\frac{t}{\tau_\phi}\right) + \exp\left(-\frac{t}{\tau_\phi^*}\right)\right],$$

where

$$\frac{1}{\tau_\phi^*} = \frac{1}{\tau_\phi} + \frac{1}{\tau_v}.$$

Therefore, if $\tau_\phi \gg \tau_v$, then both the quantum correction and negative magnetoresistance are $N$ times smaller, where $N$ is the number of valleys.

Consider now an electron with spin and the indices of $C$ as spin indices. Then

$$C^{\alpha\beta\gamma\delta}(t) = C_s(t)\delta^{\alpha\beta}\delta^{\gamma\delta}(t) + C_t(t)\sigma^{\alpha\beta}\sigma^{\gamma\delta},$$

where $C_s$ and $C_t$ are phase correlators in spin-singlet and spin-triplet channels. Again, let us first assume that there is no spin relaxation, and the spins are polarized. Then their polarization is conserved and both triplet and singlet correlators decay with the same relaxation time $\tau_\phi$. There is, however, a difference between the cases of inter-valley and spin-orbit scattering: since there are three Pauli matrices, contribution of the triplet channel is three time larger, than that of the singlet channel. In order to give two spin projections, these 3 and 1 should be subtracted:

$$C(t) = \frac{1}{2}[3C_t(t) - C_s(t)]$$

and, therefore,

$$C_s \propto \exp\left(-\frac{t}{\tau_\phi}\right); \tag{14}$$

$$C_t \propto \exp\left(-\frac{t}{\tau_\phi^*}\right), \quad \frac{1}{\tau_\phi^*} = \frac{1}{\tau_\phi} + \frac{1}{\tau_{so}}, \tag{15}$$

where $\tau_{so}$ is the time of spin relaxation due to spin-orbit interaction.

Thus, the magnetoresistance in, say, two-dimensions is given by the formula

$$\Delta\sigma^{(2)} \sim \frac{e^2}{\hbar}\int_0^{\tau_\phi} \frac{dt}{t}\left[1 - \cos\left(\frac{eDBt}{c\hbar}\right)\right]\left[\frac{3}{2}\exp\left(-\frac{t}{\tau_{so}}\right) - \frac{1}{2}\right]. \tag{16}$$

If $\tau_\phi \gg \tau_{so}$ then

$$\Delta\sigma^{(2)} \sim \frac{e^2}{\hbar}\begin{cases} -\frac{1}{2}\left(DeB\tau_\phi/c\hbar\right)^2, & \tau_\phi \ll c\hbar/DeB, \\ -\frac{1}{2}\log\left(DeB\tau_\phi/c\hbar\right), & \tau_{so} \ll c\hbar/DeB \ll \tau_\phi, \\ \log\left(DeB\tau_{so}/c\hbar\right), & \tau_{so} \gg c\hbar/DeB. \end{cases} \tag{17}$$

Therefore, the spin-orbit interaction leads to the change in the sign of magnetoresistance: positive in weak fields, it becomes negative with a twice as steep slope (in the logarithmic coordinates), as it is shown in Fig. Therefore, relatively simple measurements of magnetoresistance give a tool for determination of two relaxation times in disordered conductors: the phase breaking time $\tau_\phi$ and spin relaxation time $\tau_{so}$.

## A notion of Anderson Localization. Properties of Localized States

The study of weak localization showed that, for $d = 3$, quantum correction to conductivity remains finite provided disorder is weak enough. The relative value of this correction is determined by the ratio of the wave length of electrons at the Fermi surface $\lambda_F$ to the elastic mean free path $l$. If $l \gg \lambda_F$ (the Ioffe-Regel criterium), then the quantum correction is relatively un-important. In the case, when the Ioffe-Regel criterium vanishes, quantum effects lead to Anderson localization: quantum states decay exponentially at distances $\mathbf{r}$ larger than localization length $L_c$. As a result, if an extra density is generated at the origin at $t = 0$, then, at long enough time $t$, the averaged density distribution does not depend on time and equals to

$$\langle \rho(0,0)\rho(t,\mathbf{r})\rangle_{t\to\infty} \propto \exp\left[-\frac{|\mathbf{r}|}{L_c}\right]. \tag{18}$$

If one looks at the density of states $\mathbf{u}(E)$ of a disordered solid (see Fig 3), it has regions of localized and extended states, divided by the so called mobility edges. If electron concentration

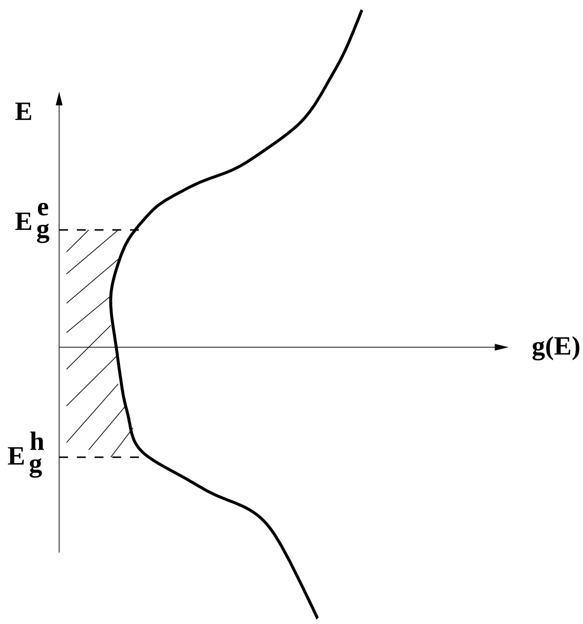


Figure 3: Density of states of a disordered semiconductor. Localizied states are scretched

changes, it corresponds to a change of the chemical potential $\mu = E_F$. If the Fermi energy belongs to the region of extended states, the material is conduction and d.c. conductivity $\sigma(0)$ at $T = 0$ is finite. If the Fermi energy belongs to the region of localized states[1], the material is the Anderson insulator and d.c. conductivity $\sigma(0)$ vanished at $T = 0$[2].
Properties of a disordered conductor are well familiar: d.c. conductivity is finite; the electron wave functions are extended, oc-

[1] This analysis is correct only if mutual interaction between electron is negligible. Otherwise, the disordered potential, properties of quantum states and many other things should be treated self-consistently

[2] For finite temperatures, conductance remains finite even in the insulating states, because electron could hop from one localized state into the other due to electron-phonon or electron-electron interaction. See lecture 15 about this, so called, hopping conductivity

cupy whole volume $V$ of a finite sample pretty uniformly and don't repel each other unless their energies $E$ and $E'$ are coming closer, than the mean level spacing $\Delta = (\nu V)^{-1}$, the density $\rho(t, \mathbf{r})$ evolves with like time the probability $W$ in Eq (2).

The properties of the Anderson insulator is determined by two factor: the density of states $\nu$ is finite, while all states are localized. This means, that two electron wave functions are well overlapping if the difference of their energies$E$ and $E'$ is larger, than $\Delta_c = (\nu L_c^d)^{-1}$. At these energy differences electron wave functions are nearly un-correlated. If the energies $E$ and $E'$ are coming closer to each other ($|E - E'| \ll \Delta_c$), the corresponding wave functions should correlate so strongly that they should be centered at long distances $|\mathbf{r} - \mathbf{r}'| \gg L_c$ from each other: otherwise an overlap of the wave functions causes level repulsion. Assuming the exponential overlap (18), we estimate this distance as

$$|\mathbf{r} - \mathbf{r}'| \sim L_c \ln\left(\frac{\Delta_c}{|E - E'|}\right). \tag{19}$$

Since the a.c.conductivity $\sigma(\omega)$ is determined by the square of the matrix element of velocity between two states $E$ and $E' = E + \hbar\omega$, we obtain the following qualitative estimate (*N.Mott*):

$$\mathrm{Re}\sigma(\omega) \sim e^2|v_\omega|^2 \frac{d^d r}{L_c^d} \sim e^2\omega^2|r_\omega|^2 \frac{|r_\omega|^{d-1}dr}{L_c^d} \sim e^2\omega^2 L_c^2 \ln^{d+1}\left(\frac{\Delta_c}{\hbar\omega}\right)^3. \tag{20}$$

This shows that the Anderson insulator differs from a conventional insulator by the singular frequency dependence of the real part of its a.c. conductivity. As for its imaginary part, it is not

[3]We assumed that $r_\omega$ is given by Eq (19) and $dr \sim L_c$

at all singular

$$\mathrm{Im}\sigma(\omega) = \frac{\omega}{2\pi}\mathrm{Re}\epsilon(0) \sim e^2\omega L_c^2. \qquad (21)$$

## Scaling Theory of Anderson Localization.

We know now about two basic phases of a disordered metal: a conductor and an Anderson insulator. Transition between these two phases occurs when the Fermi energy $E_F$ of electrons coincides with the mobility edge $E_g$. The question of how d.c.conductivity $\sigma(0)$ vanishes and how localization length $L_c$ diverges when $E_F$ approaches $E_g$ remain un-answered. We bring here an interpolating theory, which unifies description of the both conducting and insulating phases and gives a description of the transition, which is supposed to be qualitatively correct. First of all, one can notice from Eq (4) that the quantum correction $\delta\sigma$ to classical conductivity for $d = 2$ is $e^2/\hbar$ times a dimensionless function of the temperature. Therefore, 2D-conductivity measured in quantum units $e^2/\hbar$ is a dimensionless value. For an arbitrary dimension $d$, the conductance $G$ and conductivity $\sigma$ are coupled as

$$G = \sigma L^{d-2},$$

where $L$ is a typical sample size[4]. This means that:
1. For $d = 2$ conductivity and conductance (of a square shape film) are the same;
2. For an arbirary dimension $d$, the conductance $G$, measured

[4]The famous high school expression of the conductance $G$ through the length of the sample $L$ and its cross-section $S \propto L^2$ - $G = \sigma S/L$ - coinsides with this formula for $d = 3$

in quantum units $e^2/\hbar$, remains dimensionless $g = \hbar G/e^2$.
3. A good conductor has large conductance $g \gg 1$, while a bad conductor (an Anderson insulator) has a small conductance $g \ll 1$.
The quantum corrections due to interference are relatively small if dimensionless conductance is large and large in the opposite case. This all allows to conjecture the following *Scaling hypothesis*:
Let us assume that the dimensionless conductance of a sample of size (in all dimensions) $L$ is $g(L)$. Consider a sample of the size $2L$, which we might consider as composed from samples of size $L$. The law of composition could be pretty complicated. Nevertheless, we assume that the conductance $g(2L)$ of the large sample depends only on conductances $g(L)$ of its constitutes. This assumtion could be expressed in the form of the following scaling equation:

$$\frac{d\log g}{d\log L} = \beta(g). \tag{22}$$

For $d = 2$ and $g \gg 1$, Eq (4) gives the result, which coincides with that of Eq (22), if $L = L_\phi$. One can deduce from Eqs (4) and (22) that for $d = 2$ and $g \gg 1$ the scaling function $\beta(g) = -1/g + o(g^{-2})$. For an arbitrary dimension $d$ and $g \gg 1$, the conductance is $g(L) = \sigma L^{d-2} + \delta\sigma L^{d-2}$ and, therefore, $\beta(g) = d - 2 - 1/g + ...$
Since small conductances $g \ll 1$ correspond to insulating phase, $g(L) \propto \exp(-L/L_c)$ and, therefore,

$$\beta(g) = \frac{d\log g}{d\log L} = -\frac{1}{L_c}\frac{d}{d\log L}\exp(\log L) = \log g + o(1). \tag{23}$$

The scaling functions $\beta(g)$ for different dimensions $d$ is plotted against $\log g$ on Fig 4. For given strength of disorder, conductance of the sample of the size of mean free path $g(l)$ determines the initial condition for the scaling equation (22). The result of renormalization is given by solution of this equation $g(l)$ with the mentioned initial condition. For instance, Fig shows that $\beta_d(g)$ for $d = 1, 2$ and all $g$ is negative. This means that independently from degree of disorder (the value of initial condition) for large enough distances $g(L) \propto \exp(-L/L_c)$. For $d = 2$, localization length $L_c \sim l \exp(\sigma\hbar/e^2)$.
For $d > 2$, the scaling $\beta(g)$ is positive at large conductances and negative at small conductances. Therefore, it vanishes at some critical conductance $g_c$. If the initial condition $g_i$ for the scaling equation (22) $g_i = g(l)$ is larger, than the critical value $g_c$, than the scaling function is positive $\beta(g_i) > 0$, the conductance is growing with increasing the sample size $L$ and, at large

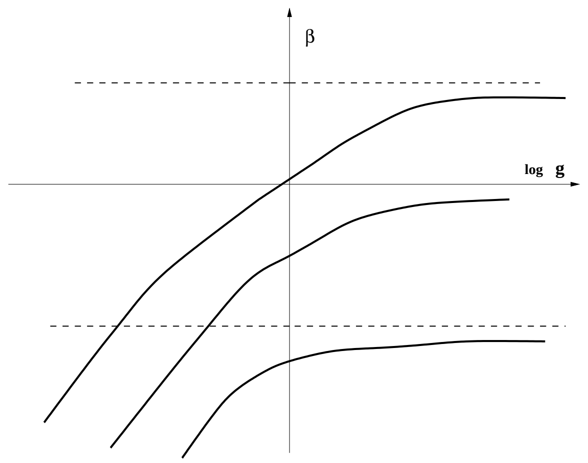


Figure 4: The Gell-Mann - Low function for localization theory

enough $L$, it follows the law $g(L) = \sigma L^{d-2}$, characteristic for a conducting phase. If the scaling function near its zero has the Taylor expansion $\beta(g) \approx (g - g_c)/sg_c$, then, for $0 < g_i - g_c \ll 1$, solution of scaling equation is

$$\log \frac{L}{l} = \int_{g_i}^{g} \frac{dg}{g\beta(g)} \approx (d-2)^{-1} \log g - s \log(g_i - g_c); \quad (24)$$

$$g(L) \propto (g_i - g_c)^{s(d-2)} L^{d-2}; \qquad \sigma \propto (g_i - g_c)^{s(d-2)}. \quad (25)$$

If the initial condition $g_i$ is smaller, than the critical value $g_c$, than the scaling function is negative $\beta(g_i) > 0$, the conductance is decreasing with increasing the sample size $L$ and, at large enough $L$, it follows the law $g(L) \propto \exp(-L/L_C)$, characteristic for an insulating phase. For $0 < g_c - g_i \ll 1$, solution of scaling equation is

$$\log \frac{L}{l} = \int_{g}^{g_i} \frac{dg}{g\beta(g)} \approx s \log(g_i - g_c) - \log \log g; \qquad (26)$$

$$g(L) \propto \exp\left[-\frac{L}{L_c}\right]; \qquad L_c \propto (g_i - g_c)^{-s(d-2)}. \qquad (27)$$

It is natural to assume that $g_i - g_c$ varies linearly while the Fermi energy $E_F$ passes through the mobility edge $E_g$ ($g_i - g_c \propto E_F - E_g$). This assumption leads us to the following laws at the Anderson transition:

$$\sigma \propto (E_F - E_g)^{s(d-2)}, \ E_F > E_g; \qquad L_c \propto (E_g - E_F)^{-s}, \ E_F < E_g. \quad (28)$$

Thus, the scaling hypothesis predicts vanishing of d.c. conductivity $\sigma$ at the Anderson transition as a power of $E_F - E_g$ and divergence of localization length $L_c$ as the same (but negative) power.