Lecture 10.
Superconductivity. Ginzburg-Landau Theory. Type II Superconductivity. Josephson Effect

Superconductivity was discovered in 1911 in Leiden by Kammerling-Onnes, who observed vanishing the resistivity in mercury below critical temperature $T_c = 4.15$ K. The further studies in Leiden shown that superconductivity is destroyed by magnetic field. Switching on the magnetic field generate in a superconducting ring current, which persists in the ring as long as the temperature T remains below T_c. A new significant step in studies of this phenomena was made in 1932 in Berlin by Meissner and Ochsenfeld, who proved that the transition from a normal to superconducting state in magnetic field is a reversible transition between two thermodynamic phases. This implied application of the theory of phase transition to superconductivity (*V.L. Ginzburg and L.D. Landau 1950*).

Spontaneous symmetry breaking. Second order phase transition. Ginzburg-Landau theory

Let us begin with an assumption that superconductivity occurs as a result of the second order transition. Superconducting phase corresponds to spontaneous breaking of the global gauge invariance and is characterized by the complex order parameter $\Psi(\mathbf{r})$. According to Landau's theory, the free energy F_s of superconducting phase is given by the expansion:

$$F_s - F_n = \left[\frac{\alpha}{2}(T - T_c)|\Psi|^2 + \frac{\beta}{4}|\Psi|^4\right] V \tag{1}$$

where F_n is the free energy of the normal phase and V is the volume of the sample. If the order parameter varies in space, the volume V in Eq (1) is replaced by the integral $\int d\mathbf{r}$. An extra contribution to free energy should take care of the finite gradients of the order parameter.

$$F_\nabla = \int d\mathbf{r} \frac{D}{2} \left| \left(i\nabla - \frac{e\mathbf{A}}{c\hbar} \right) \Psi \right|^2 .^1 \tag{2}$$

This term shows that the uniform order parameter corresponds to the minimum of the free energy F. The vector potential of magnetic field $\mathbf{A}$ should be added in order to preserve the local gauge invariance. The energy of magnetic field $\mathbf{B} = \text{curl } \mathbf{A}$,

$$\int d\mathbf{r} \, \frac{\mathbf{B}^2}{8\pi} = \int d\mathbf{r} \, \frac{(\text{curl } \mathbf{A})^2}{8\pi}$$

should also be added to free energy. Finally, in the presence of external field $\mathbf{H}$, the Legendre transform to the Gibbs potential $G = F - \mathbf{BH}/4\pi$ should be made. Therefore, the Gibbs potential looks like

$$G_s - G_n = \int d\mathbf{r} \left\{ \frac{\alpha(T - T_c)}{2} |\Psi|^2 + \frac{\beta}{4}|\Psi|^4 \right\} +$$
$$+ \int d\mathbf{r} \left\{ \frac{D}{2} \left| \left(i\nabla - \frac{e\mathbf{A}}{c\hbar} \right) \Psi \right|^2 + \frac{(\text{curl } \mathbf{A})^2}{8\pi} - \frac{\mathbf{H} \text{ curl } \mathbf{A}}{4\pi} \right\} . \tag{3}$$

[1] In the next lecture we consider the BCS microscopic theory of superconductivity and learn that the supercurrent is driven by the Cooper pairs, which have charge $2e$. So, all formulae in this lecture should be corrected by the replacement $e \to 2e$.

In order to find the spatial distribution of magnetic field $\mathbf{B}$ and the order parameter Ψ, one have to minimize the Gibbs potential. Varying it with respect to $\Psi^*(\mathbf{r})$, we obtain two contributions: from the bulk and from the surface. Since the equilibrium configuration corresponds to a minimum of G, these contributions both vanish. This gives the equation

$$D\left[\left(i\nabla - \frac{e\mathbf{A}}{c\hbar}\right)\right]^2 \Psi + \alpha(T - T_c)\Psi + \beta|\Psi|^2\Psi = 0 \tag{4}$$

with the boundary condition

$$\mathbf{n}\left(i\nabla - \frac{e\mathbf{A}}{c\hbar}\right)\Psi = 0 \tag{5}$$

Variation of the Gibbs' potential (3) with respect to $\mathbf{A}$ leads to the equation of magnetostatics

$$\text{curl curl } \mathbf{A} = \frac{4\pi\mathbf{j}}{c} \tag{6}$$

$$\mathbf{j} = ieD\left(\Psi^*\nabla\Psi - \nabla\Psi^*\Psi + 2\frac{e\mathbf{A}}{c\hbar}\Psi^*\Psi\right) \tag{7}$$

and a standard boundary condition for tangential components of magnetic field

$$(\text{curl } \mathbf{A})_t = \mathbf{H}_t \tag{8}$$

The system (3), (5) and (7) with the boundary conditions (4) and (8) define a mathematical problem of the theory of superconductivity.

Meissner Effect, London Penetration Depth

First of all, let us introduce the order parameter in the absence of gradients:

$$\Psi_0 = \sqrt{\frac{\alpha(T - T_c)}{\beta}}$$

Then, introducing a new variable

$$\Psi = \psi \Psi_0, \tag{9}$$

the equations can be rewritten as

$$-\psi + |\psi|^2\psi - \xi^2(\nabla + i\frac{e\mathbf{A}}{c\hbar})^2\psi = 0 \tag{10}$$

$$\text{curl curl } \mathbf{A} = -\frac{1}{\lambda_L^2}\left(\frac{i\Phi_0}{4\pi}\left(\psi^*\nabla\psi - \nabla\psi^*\psi\right) + \mathbf{A}\psi^*\psi\right), \tag{11}$$

where

$$\xi = \sqrt{\frac{D}{\alpha(T_c - T)}} \tag{12}$$

If one assumes that

$$\psi(\mathbf{r}) = \exp\left[i\phi(\mathbf{r})\right],$$

then

$$\text{curl curl } \mathbf{A} = -\frac{1}{\lambda_L^2}\left(\frac{\Phi_0}{2\pi}\nabla\phi + \mathbf{A}\right). \tag{13}$$

Acting by the operator curl on both sides of Eq (13), assuming the gauge div $\mathbf{B} = 0$ and recalling that curl $\mathbf{A} = \mathbf{B}$, we obtain Londons' equation:

$$\nabla^2 \, \mathbf{B} = \frac{\mathbf{B}}{\lambda_L^2}. \tag{14}$$

Thus, the magnetic field decays with the depth z of a superconductor as $\exp(-z/\lambda_L)$, where λ_L is the penetration depth.

Critical magnetic field parallel to a superconducting film.

Consider a very thin film of thickness d, parallel to the x, y-plane, in a parallel magnetic field $\mathbf{B}$, pointing in the positive y-direction. The vector-potential $\mathbf{A}$ may be chosen in the form

$$A_y = A_z = 0, \quad A_x = Bz.$$

For this gauge, the order parameter $\psi(x, y, z)$ has the form

$$\psi(x, y, z) = \phi(z) \exp[i(p_x x + p_y y], \quad -\frac{d}{2} < z < \frac{d}{2}.$$

The gradient term of the Ginzburg-Landau free energy per unit volume looks now as

$$F_{\nabla} = \frac{D}{2d} \int dz \left[p_y^2 \left|\phi(z)\right|^2 + \left| \left(\frac{d}{dz} - i\frac{eBz}{c\hbar} \right) \phi(z) \right|^2 \right] \tag{15}$$

In the leading approximation in magnetic field

$$\phi(z) = \text{const}, \quad p_y = 0$$

and the term containing magnetic field can be treated by the means of perturbation theory. This gives the quadratic term in the Ginzburg-Landau free energy in the form

$$F_{\text{quadr}} = \frac{1}{2} \left(\frac{De^2 B^2 d^2}{c^2 \hbar^2} + \alpha(T - T_c) \right) \left|\phi\right|^2. \tag{16}$$

Eq (16) shows that a parallel magnetic field B causes a destruction of superconductivity even at $T < T_c$ if $B > B_c$, where

$$B_c = \left(\frac{c\hbar}{ed} \right) \left[\frac{\alpha(T_c - T)}{D} \right]^{1/2}. \tag{17}$$

The relation (17) can be expressed in the form of the following estimate

$$\xi \sim \frac{L_H^2(B_c)}{d}, \qquad L_H(B) = \sqrt{\frac{c\hbar}{eB}}. \tag{18}$$

Eqs (17) and (18) are valid if $d \ll L_H$, and if the film thickness d is smaller, than the correlation length ξ,

$$d \ll \xi.$$

Thus, there are two parameters of dimension of length, which characterize spatial variation in superconductors: the London penetration depth λ_L for screening of magnetic field and correlation length of superconductivity ξ for the variation of superconducting order parameter ψ. Depending on the ratio of these two lengths, one has to distinguish between the type I superconductors ($\lambda_L \ll \xi$) and the type II superconductors ($\lambda_L \gg \xi$).

Type I superconductors. Intermediate state. Surface energy of NS Boundary

As one can see from Eq (3), if the magnetic field $\mathbf{B}$ is pushed out of the volume of a superconductor, this adds and extra energy $\mathbf{B}^2/8\pi$ to the free energy of the superconductor unit volume. When this extra energy exceeds the free energy difference $\Delta F = F_n - F_s$, the sample transit into the normal state. This corresponds to the thermodynamic critical field B_c.

$$B_c = \sqrt{8\pi(F_n - F_s)}, \tag{19}$$

Consider a wide superconducting plate of thickness h in a perpendicular magnetic field $\mathbf{B}$. In order to minimize the energy,

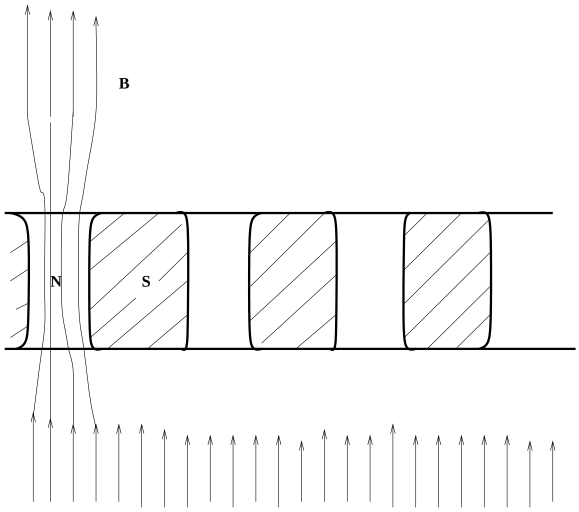

Figure 1: Intermediate state in a superconducting plate in a perpendicular magnetic field field

the superconducting order parameter is zero in some areas of the sample, giving way to magnetic flux, and non-zero in another areas. A layered structure of superconducting and normal stripes of widths a_n and a_s respectively is formed (see Fig 1). We will see that $a \sim a_s \sim a_n \ll h$ and, therefore, the ns-boundary is almost a plane, parallel to the direction of the magnetic field. At the ns-boundary the magnetic field is equal to B_c and it remains the same in the whole normal region. Since the magnetic flux is conserved,

$$B(a_s + a_n) = B_c a_n; \quad \frac{a_s}{a_n} = \frac{B_c - B}{B}$$

Thus, the bulk energies in both s and n domains are equal, and the balance of magnetostatic and surface energies determines the sizes of domains a. Magnetostatic energy is related to the redistribution of magnetic field in the area of linear size of the order of a. The energy per single domain is of the order of $B_c^2 a^2 l$, where l is the dimension in the direction, perpendicular to the plane of Fig 1. The surface energy per single domain is $\sigma_{ns} hl$, where σ_{ns} is the surface energy per unit are. Since the number of domains per unit transverse length is $N = 1/a$, the total free energy of the layered structure is

$$\mathcal{F} = B_c^2 al + \frac{\sigma_{ns} hl}{a}. \tag{20}$$

The minimum of this free energy is reached at the equilibrium value of a, which is equal to

$$a = \sqrt{\frac{\sigma_{ns} h}{B_c^2}} \sim \sqrt{h\delta}, \quad \delta = \frac{\sigma_{ns}}{B_c^2}. \tag{21}$$

The physical meaning of δ is the thickness of the ns-boundary. The profiles of the field $B(x)$ and the order parameter $\psi(x)$ at the ns-boundary is given by the Ginzburg-Landau and Maxwell equations (4)-(7) and shown in Fig 2. The value of the surface energy σ_{sn} is given by the extra contribution to the Gibbs

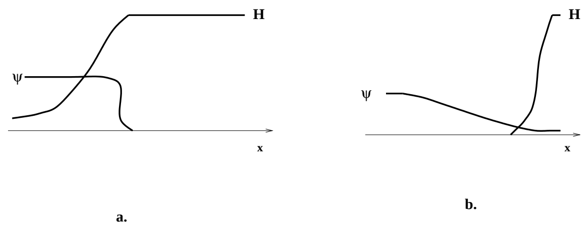

Figure 2: Dependence of the order parameter ψ and magnetic field H on transverse coordinate near the NS-boundary for type I (a/.) and type II (b/.) superconductors

potential G.

$$\sigma_{ns} = \int_{-\infty}^{+\infty} dx[G(x) - G_s] = \frac{B_c^2}{8\pi}(-\lambda_L + \sqrt{2}\xi). \tag{22}$$

This implies that the surface energy σ_{ns} is positive if $\lambda_L < \sqrt{2}\xi$ (type I superconductors). Under these conditions, the free energy minimum corresponds to the minimum of the surface area of the ns-boundary. This prevent the ns-boundary from meandering and described layered structure is stable. Under the the opposite condition (type II superconductors), the surface energy is negative, which, naturally, creates some problems.

Type II superconductivity. Vortices. Lower critical field.

If the surface energy is negative, the normal and superconducting phases cannot not remain in a contact, because the boundary meandering leads to lower energy. The result is that the penetration of magnetic flux into the superconductor occurs in the form of strings, along which the order parameter vanishes. Since superconductivity is destroyed at the centre of this string, the magnetic field can penetrate into the superconducting body along the string. If $\xi \ll \lambda_L$, superconductivity is destroyed in a relatively narrow region near the string, the modulus of the order parameter remains a constant in the rest of the sample. As for the phase ϕ of the order parameter, it acquires 2π while azimuthal angle θ make a complete circle around the string. Since the problem is azimuthally symmetric, this means that

$$\phi = \theta + \text{const} \tag{23}$$

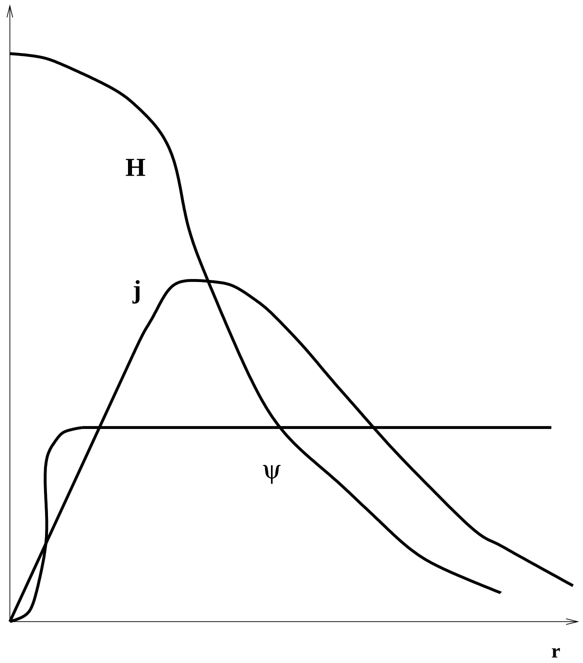

Figure 3: Dependence of the order parameter ψ, ciurrent j and magnetic field H on distance r from the axis of the vortex

The current $\mathbf{j}$ is proportional to the gradient of the phase and, therefore, circulate around this string, which forms a superconducting vortex (Abrikosov vortex).
Acting by curl on both sides of Eq (13), we obtain the equation for the magnetic field $\mathbf{B}$ around the vortex:

$$\mathbf{B} + \frac{1}{\lambda_L^2}\text{ curl curl }\mathbf{B} = \frac{\Phi_0}{2\pi}\text{curl }\nabla\phi = \Phi_0\ \delta(\mathbf{r}_\perp)^2. \qquad (24)$$

The right-hand side of Eq (24) shows, in particular, that the total magnetic flux, associated with the single vortex, is equal to

[2]The second part of Eq(24) can be obtained by integrating both its sides over a small area near origin

the flux quantum Φ_0. The solution of equation for the magnetic field $\mathbf{B}$ has the following form

$$B = \frac{\Phi_0}{2\pi\lambda_L^2} K_0\left(\frac{r}{\lambda_L}\right), \quad K(x) = \begin{cases} \ln(1/x), & x \ll 1; \\ x^{-1/2}e^{-x}, & x \gg 1. \end{cases}, \tag{25}$$

where $K_0(x)$ is the MacDonald function of the zeroth order. The profiles for the magnetic field $\mathbf{B}$, the current density $\mathbf{j} \propto \mathrm{curl}\,\mathbf{B}$ and the modulus of the order parameter $|\psi|$ are presented in Fig 3.

The extra energy ϵ per the vortex unit length, which is required for creation of this defect of the long range order, is

$$\epsilon = \frac{1}{8\pi}\int d\mathbf{r}_\perp \left[\mathbf{B}^2 + \lambda_L^2 \left(\mathrm{curl}\ \mathbf{B}\right)^2\right] \tag{26}$$

Using Eq (25), we obtain

$$\epsilon = \frac{\Phi_0}{8\pi} B(0) = \left(\frac{\Phi_0}{4\pi\lambda_L}\right)^2 \ln\left(\frac{\lambda_L}{\xi}\right). \tag{27}$$

Since the vortex has a finite energy, it cannot be created in a weak field because it costs the energy. Therefore, the Meissner effect remains until the external magnetic field exceeds the lower critical field H_{c1}. If the external magnetic field $\mathbf{H}_0$ is finite, a demagnetization energy

$$-\frac{\mathbf{B}\mathbf{H}_0}{4\pi}$$

can make a strong negative contribution to the total Gibbs' potential of the vortex and make creation of the vortices favorable. The field at which this compensation occurs is called the lower critical field H_{c1}:

$$H_{c1} = \frac{4\pi\epsilon}{\Phi_0} = \frac{\Phi_0}{4\pi\lambda_L^2}\left[\ln\left(\frac{\lambda_L}{\xi}\right) + .08\right]. \tag{28}$$

For $H > H_{c1}$, magnetic flux starts to penetrate into superconductor, and diamagnetic momentum decreases with increasing megnetic field H_0.

Mixed state. Upper critical field

Thus, the magnetization curve for the bulk type II superconductors looks like that shown in Fig 4. For $H < H_{c1}$, the Meissner effect leads to ideal diamagnetism. For $H > H_{c1}$, the vortices begin to penetrate into superconductors, forming a new, mixed state, while the Meissner phase becomes metastable [3]. For $H_{c1} < H < H_{c1} \ln(\lambda_L/\xi)$, the vortices stay well apart, overlapping only by exponential tails. The penetration of magnetic field is heterogeneous. For $H \gg H_{c1} \ln(\lambda_L/\xi)$, the

[3] Thermodynamic critical field $H_c \sim \Phi_0/\xi\lambda_L$ corresponds to the limit of stability of the Meissner phase.

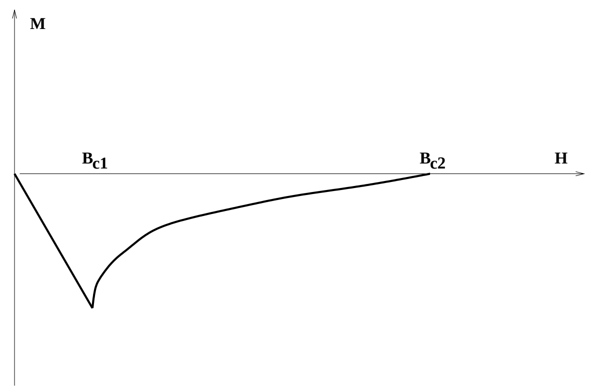

Figure 4: Magnetization curve in type II superconductors

vortices are strongly overlapped, which leads to almost homogeneous penetration of magnetic flux into the superconductor. The diamagnetic magnetization M remains significant, but it goes down, when the magnetic field increases, to disappear in a strong enough magnetic field H_{c2}, the upper critical field.
To find the upper critical field, we must consider instability with respect to the superconducting transition in the presence of the magnetic field. Looking at the linear in ψ part of Eq (17) and recalling the spectrum of the Schroedinger operator in magnetic field, we find that

$$-1 + \xi^2 \frac{eH_{c2}}{c\hbar} = 0, \qquad H_{c2} = \frac{\Phi_0}{2\pi\xi^2}. \tag{29}$$

This shows that superconductivity dies out in the type two superconductors only when Abrikosov's vortices come so close to each other, that their normal cores overlap[4].

Josephson Effect

Consider two bulk superconductors, coupled by a weak link. In the leading approximation, neglecting mutual interaction between superconductors, the order parameter $\Psi_{1,2}$ in both superconductors have non-correlated phases:

$$\Psi_{1,2} = \Psi_0 e^{i\chi_{1,2}}.$$

In the following approximation in the coupling between two superconductors, their free energy consists of a terms which depends difference of the phases $\phi = \chi_1 - \chi_2$ of the phases of the

[4]The normal core of the vortex has a radius of the order of the correlation length ξ.

order parameters $\Psi_{1,2}$:

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 - \frac{K_s}{2}\left(\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1\right) = \mathcal{F}_1 + \mathcal{F}_2 - \frac{\Phi_0 J_c}{c}\cos\phi. \quad (30)$$

At the equilibrium, the order parameters $\Psi_{1,2}$ in the bulk superconductors has identical phases ($\phi = 0$). This means that free energy (30) has a minimum at $\phi = 0$, i.e. $J_c > 0$. Since the interaction of magnetic field with the current is given by the extra term $\delta\mathcal{H} = \mathbf{j}\mathbf{A}/c$ in the Hamiltonian, the super-current through weak link is connected with the derivative

$$J_s = \frac{\partial \mathcal{F}}{\partial \phi} = J_c \sin\phi. \quad (31)$$

This expression may be obtained, if a ring with a weak link and a flux Φ through the hole is considered. Thus, is a phase difference of the order parameters is kept $-\pi/2 < \phi < \pi/2$, a dissipativeless current is flowing through the weak link. Its maximal value is J_c. On the other hand, if a current $-J_c < J < J_c$ is driven through the weak link, this current may flow without dissipation, which means that no voltage V occurs across the junction. This is why J_c is called "the critic current" of a Josephson junction.

If the current $J > J_c$ is driven through the weak link, a finite voltage V across the junction occurs. Since the weak link has a finite resistance R, a finite voltage V results in a finite current of excitations $J_n = V/R$. Thus, we have an equation for total current J:

$$J = J_s + J_n = J_c \sin\phi + \frac{V}{R}. \quad (32)$$

The second equation couples the phase difference ϕ and the voltage V. Indeed, since two superconductors have different chemical potentials

$$\mu_1 = \mu_2 + eV,$$

the phase difference $\phi = \chi_1 - \chi_2$ varies with time:

$$\hbar\,\frac{d\phi}{dt} = eV. \qquad (33)$$

Equations (32) and (33) define time evolution of both voltage $V(t)$ and phase $\phi(t)$ for given current J. Solving these equations we find dependence of time averaged $\langle V\rangle$ voltage on current (current-voltage characteristics):

$$J^2 + J_c^2 = \frac{\langle V\rangle^2}{R^2}, \quad V(t) = \frac{\langle V\rangle}{1+\cos\omega t}, \quad \omega = \frac{e\langle V\rangle}{\hbar}. \qquad (34)$$

We see also from Eq (34) that the voltage has a periodic component, which frequency ω is coupled with the d.c. voltage $\langle V\rangle$ by the relation, which contains only fundamental constants. The Josephson junction is a non-linear element of the electric circuit. So, we may expect interesting non-linear properties (Shapiro steps), Josephson radiation on fundamental frequency $\omega = eV/\hbar$ and of electric properties of the circuit on magnetic field. All this can be found in the problems.