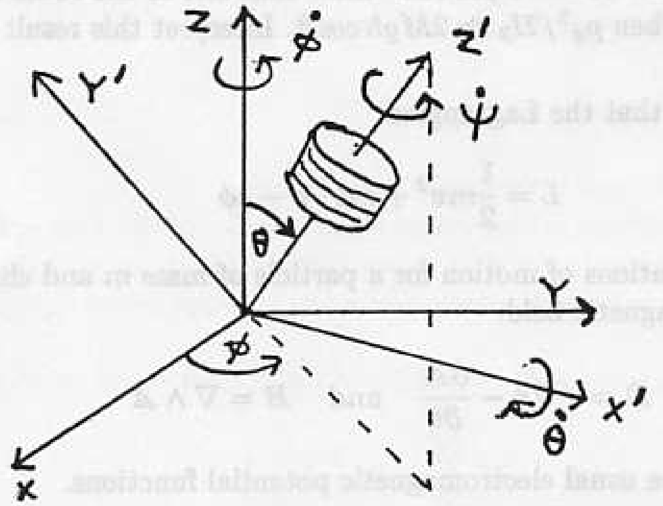
NATURAL SCIENCES TRIPOS Part II

Wednesday 13 January 1999 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **3** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 The figure shows a symmetrical spinning top pivoted at the origin:



The axes $[x, y, z]$ are fixed in space with z vertical. The axes $[x', y', z']$ are defined with z' along the axis of symmetry of the top, and y' is perpendicular to z' and is in the plane of z and z'. The top is spinning about z' with angular velocity $\dot{\psi}$ and precessing about the z axis with angular velocity $\dot{\phi}$. There is a further component of angular velocity $\dot{\theta}$ about the x' axis.

Taking (θ, ϕ, ψ) as generalised coordinates, the Lagrangian for the top may be written as:

$$L = \frac{1}{2}I_2\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_1\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta$$

where I_1 and I_2 are the moments of inertia of the top parallel and perpendicular to z', M is the mass of the top and h is the distance to its centre of mass along z'.

(TURN OVER for continuation of question 1

(a) Explain carefully the meaning of each term in the expression for L. [4]

(b) Show that the corresponding canonical momenta p_ϕ and p_ψ are constant and write down explicit expressions for p_ϕ and p_ψ. [4]

(c) Write down an expression for E, the total energy of the system. It may be assumed that E is a third constant of the motion. [2]

(d) Hence show that

$$\frac{1}{2}I_2\dot{\theta}^2 = K - V_{\text{eff}}(\theta)$$

where K is a constant of the motion and V_{eff} is a function of θ. [4]

(e) Explain why $\frac{\partial V_{\text{eff}}}{\partial \theta} = 0$ is a necessary condition for steady motion with $\theta =$ constant. [4]

(f) Show that this condition leads to a quadratic equation for $\dot{\phi}$ assuming $\theta \neq 0$ or π. [8]

(g) Find the roots of this equation. Show that one of the roots tends to the limit mgh/p_ψ when $p_\psi{}^2/2I_2 \gg 2Mgh\cos\theta$. Interpret this result physically. [7]

2 Show explicitly that the Lagrangian:

$$L = \frac{1}{2}m\boldsymbol{v}^2 + e\boldsymbol{A}\cdot\boldsymbol{v} - e\phi$$

yields the correct equations of motion for a particle of mass m and charge e moving in a electromagnetic field:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad \textit{and} \quad \boldsymbol{B} = \nabla\wedge\boldsymbol{A}$$

where $\boldsymbol{A}$ and ϕ are the usual electromagnetic potential functions. [10]

For the case of cylindrical symmetry where $\phi = 0$ and $\boldsymbol{A} = (0, rf(z), 0)$ using cylindrical polar coordinates (r, θ, z) and where $f(z)$ is a function of z:

(a) Write down the equations of motion for the particle. [6]

(b) Verify that the total energy of the particle is a constant of the motion. [7]

(c) Show that the Lagrange equation for θ gives rise to a second constant of the motion. [5]

(d) What are the conditions required for the particle travel in an orbit with $r =$ constant? [5]

(TURN OVER

3 Write down Hamilton's equations for a particle described by generalized coordinates $[q_1, q_2, q_3]$ and momenta $[p_1, p_2, p_3]$. [4]

Describe the concept of *phase space* associated with these six parameters. [4]

State and prove Liouville's theorem for the phase space density of an ensemble of such particles. [7]

Discuss applications of Liouville's theorem, with examples from three areas of physics. [18]

4 A charged particle radiates energy at a rate proportional to the square of its acceleration, $\ddot{x}^2$. Show that for *periodic* motion this is equivalent to the action of a force $+\dddot{x}$ acting on the particle, and so the equation of motion for radiation damped simple harmonic oscillator might be

$$\ddot{x} + x - \varepsilon\,\dddot{x} = f(t)$$

where $f(t)$ is the external driving force.

(a) Describe the method of solving this equation in the Fourier form $x(\omega) = G(\omega)\,f(\omega)$ and seeking a causal linear response function $G(t) = \int G(\omega)\,e^{-i\omega t}\,\frac{d\omega}{2\pi}$ [9]

(b) Show that the poles for $\varepsilon = 0$ give a sensibly damped response for small ε. [12]

(c) Show that for small $\varepsilon > 0$ the third pole which arises is pathological. What has gone wrong in logical scheme of this formulation? [12]

(Treat the cubic equation at $\varepsilon \ll 1$ with the help of perturbation, thus finding approximate corrections to two quadratic roots; then the estimate for the remaining (new) root).

5 A charged Brownian particle diffuses on a (x-y) plane in a constant external electric field E. Write down the Langevin equations and explain the limit of overdamped motion. [13]

In the overdamped limit, find the long-time average characteristics of this motion on a plane. [20]

6 Give an account of Langevin approach to stochastic processes. [10]

Explain the concepts and approximations involved in the derivation of the diffusion equation. [13]

Illustrate the role of the diffusion equation in statistical and quantum mechanics. [10]