UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

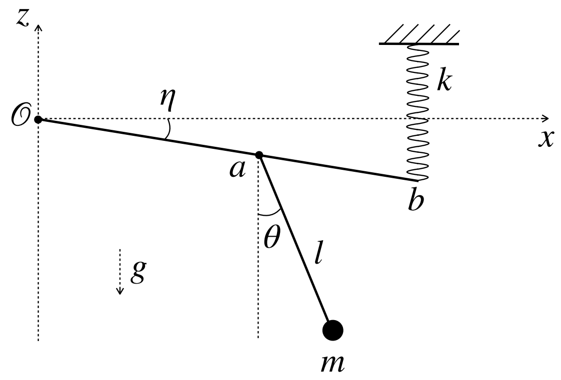
Friday 20 January 2023 2 pm to 4 pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains* **11** *sides, including this one.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 An end $\mathcal{O}$ of a massless rod of length b is fixed, allowing the rod to swing by an angle η in the vertical $x - z$ plane, as shown in the figure below. The other end of the rod is connected to a support by a spring of force constant k. The spring is sufficiently long that it can be considered to remain vertical at all times. At the natural spring length, the rod makes an angle η_0 with the x axis. The rod is connected, at a distance a from $\mathcal{O}$, to one end of a second massless rod of length l that is allowed to swing freely, making an angle θ with the vertical. A bob with a mass m is attached to the free end of the second rod.



(a) Show that the lagrangian of the system is given by

$$\begin{aligned} L &= \frac{1}{2}m\left[a^2\dot{\eta}^2 - 2al\sin(\eta+\theta)\dot{\eta}\dot{\theta} + l^2\dot{\theta}^2\right] \\ &+ mg(l\cos\theta + a\sin\eta) - \frac{1}{2}b^2k(\sin\eta - \sin\eta_0)^2. \end{aligned}$$

[6]

[Partly seen] In x, y coordinates the lagrangian reads

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz - \frac{1}{2}k\Delta^2$$

where the bob's coordinates are

$$x = a\cos(\eta) + l\sin(\theta), \quad z = -l\cos(\theta) - a\sin(\eta)$$

Hence the kinetic part of the lagrangian is

$$T = \frac{1}{2}m\left[a^2\dot{\eta}^2 - 2al\sin(\eta+\theta)\dot{\eta}\dot{\theta} + l^2\dot{\theta}^2\right]$$

while the rest is given by

$$mg(l\cos(\theta) + a\sin(\eta)) - \frac{1}{2}b^2k(\sin(\eta) - \sin(\eta_0))^2$$

(b) Find the Euler-Lagrange equations of motion. [6]

[Partly seen] The Euler-Lagrange equations read $(d/dt)(\partial L/\partial \dot{q}_i) = \partial L/\partial q_i$. For the θ angle the equation is

$$m\left[l^2\ddot{\theta} - al\sin(\eta+\theta)\ddot{\eta} - al\cos(\eta+\theta)\dot{\eta}(\dot{\eta}+\dot{\theta})\right]$$
$$= -mgl\sin(\theta) - mla\cos(\eta+\theta)\dot{\eta}\dot{\theta}$$

and for the η angle it is

$$m\left[a^2\ddot{\eta} - al\sin(\eta+\theta)\ddot{\theta} - al\cos(\eta+\theta)\dot{\theta}(\dot{\eta}+\dot{\theta})\right]$$
$$= \cos(\eta)(mga - b^2k(\sin(\eta) - \sin(\eta_0))) - mla\cos(\eta+\theta)\dot{\eta}\dot{\theta}$$

(c) Assuming that the spring can freely intersect with the rods and the bob, find all equilibrium positions of the system. [5]

[Unseen] In equilibrium the equations of motion become

$$\begin{aligned} 0 &= -mgl\sin(\theta) \\ 0 &= \cos(\eta)(mga - b^2k(\sin(\eta) - \sin(\eta_0))) \end{aligned}$$

which is solved by $\sin\theta = 0$, and $\cos(\eta) = 0$ or $\sin(\eta) = \sin(\eta_0) + mga/b^2k$.

Hence, in equilibrium θ can take values 0 and π. At the same time for η we can have $\eta = \pm\pi/2$, or $\eta = \arcsin(\sin(\eta_0) + mga/b^2k), \pi - \arcsin(\sin(\eta_0) + mga/b^2k)$.

(d) Find the value of η_0 so that the first rod has an equilibrium position with $\eta = 0$ and find the values of the other constants for which such an equilibrium is possible. [3]

[Unseen] We need to enforce the condition

$$mga - b^2k(\sin(0) - \sin(\eta_0)) = 0$$

which implies

$$\sin(\eta_0) = -mga/b^2k.$$

Evidently this is only possible if $mga/b^2k \leq 1$.

(e) Consider an equilibrium position with $\theta = \eta = 0$. By analysing small fluctuations, determine whether the equilibrium is stable or unstable. [4]

[Partly seen] The linearized equations of motion read

$$\begin{aligned} ml^2\ddot{\theta} &= -mgl\theta \\ ma^2\ddot{\eta} &= (mga - b^2k\eta + b^2k\sin(\eta_0)) = -b^2k\eta, \end{aligned}$$

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where in the last step we used the condition

$$\sin(\eta_0) = -mga/b^2k.$$

The solution can be found in the form $\theta = \mathrm{Re}\,\theta_0 e^{i\omega_\theta}, \eta = \mathrm{Re}\,\eta_0 e^{i\omega_\eta}$ with

$$\omega_\theta^2 = g/l, \quad \omega_\eta^2 = (b^2/a^2)(k/m)$$

All frequencies are real, so it is a stable equilibrium.

(f) Discuss the time-dependence of the lagrangian and its implications for the conserved quantities in this system. [1]

[**Partly seen**] The lagrangian contains no explicit time-dependence and therefore the total energy, i.e. the Hamiltonian , has to be conserved.

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2 (a) State Noether's theorem for a complex scalar field ϕ whose dynamics is described by a lagrangian density $\mathcal{L}$ and derive the form of the conserved current J^μ. [4]

[**Bookwork**] According to Noether's theorem, each continuous symmetry of a lagrangian corresponds to a conserved current J^μ such that $\partial^\mu J_\mu = 0$. For a complex scalar field, the shift in the lagrangian density is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\mu\delta\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(\partial_\mu\delta\phi^*).$$

Applying the Euler-Lagrange equations $\partial\mathcal{L}/\partial\phi = \partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\phi))$ and $\partial\mathcal{L}/\partial\phi^* = \partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\phi^*))$ we obtain

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*\right]$$

so if $\delta\mathcal{L} = 0$ we have a conserved current given by $J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\delta\phi^*$. If instead $\delta\mathcal{L}$ shifts by a total derivative, such that $\delta\mathcal{L} = \partial_\mu K^\mu$, then the current shifts by $J^\mu \to J^\mu - K^\mu$.

Consider the lagrangian density

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* - \epsilon(\phi + \phi^*)^2 - \lambda(\phi\phi^*)^2.$$

(b) Assuming $\epsilon = 0$, find a continuous symmetry transformation that acts on ϕ but not on the spacetime coordinates and find the associated conserved current

J^μ. [3]

[Bookwork] For $\epsilon = 0$, the given lagrangian density has a phase rotation symmetry $\phi \to e^{i\alpha}\phi$, with a constant α. The corresponding conserved current can be found by first computing the infinitesimal form of the symmetry transformation $\phi \to \phi + \delta\phi, \phi^* \to \phi + \delta\phi^*$ with $\delta\phi = i\alpha\phi, \delta\phi^* = -i\alpha\phi^*$. The current can be found, up to a constant overall factor as

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* = i\alpha(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi).$$

(c) Now assume $\epsilon \neq 0$. Does the previously identified transformation remain a symmetry of the lagrangian density? If not, show how the lagrangian density transforms under an infinitesimal version of the same transformation. Are there any continuous or discrete symmetries that remain when $\epsilon \neq 0$? [3]

[Unseen] The symmetry considered in the previous part is not a symmetry any more. Computing the lagrangian density variation explicitly we find

$$\delta\mathcal{L} = -2\epsilon i\alpha(\phi^2 - \phi^{*2}).$$

There remains an unbroken discrete $\phi \to -\phi$ symmetry and we of course have Poincarè invariance (namely the group of Lorentz transformations and spacetime translations).

(d) By modifying your derivation in (a), find how the continuity equation $\partial_\mu J^\mu = 0$ changes when $\epsilon \neq 0$. [3]

[Partly seen] Computing the lagrangian density variation explicitly we find

$$\delta\mathcal{L} = -2\epsilon i\alpha(\phi^2 - \phi^{*2})$$

hence (taking account of the dropped constant in the definition of J^μ)

$$\partial_\mu J^\mu = -2\epsilon(\phi^2 - \phi^{*2}).$$

For the remainder of this question, assume that $m^2 < 0$, $\epsilon < 0$, and $\lambda > 0$.

(e) Use the parameterisation $\phi = a + ib$, where a, b are real scalar fields, to find all possible values of ϕ where the system is in its ground state. [4]

[Partly seen] The potential reads

$$V = m^2(a^2 + b^2) + 4\epsilon a^2 + \lambda(a^2 + b^2)^2.$$

The first derivatives of the potential read

$$V'_a = 2a(m^2 + 4\epsilon + 2\lambda(a^2 + b^2)) = 0$$

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$$V_b' = 2b(m^2 + 2\lambda(a^2 + b^2)) = 0$$

This has solutions for $(a \neq 0, b = 0)$, $(a = 0, b \neq 0)$, $(a = 0, b = 0)$. (Note that there are no solutions with $(a \neq 0, b \neq 0)$.)

The case $(a \neq 0, b = 0)$ leads to $a^2 = -(m^2 + 4\epsilon)/(2\lambda)$.

The case $(a = 0, b \neq 0)$ leads to $b^2 = -(m^2)/(2\lambda)$.

The two deepest extrema (and hence the minima of the potential) are at $b = 0, a = \pm\sqrt{-(m^2 + 4\epsilon)/(2\lambda)} \implies \phi = \pm\sqrt{-(m^2 + 4\epsilon)/(2\lambda)}$. The degeneracy of the two minima is the result of the unbroken discrete symmetry mentioned above.

(f) Use the parameterisation $\phi = (v + \sigma)e^{i\theta}$ to find the masses of the real scalar fields σ and θ in the ground state $\phi = v$ by expanding $\mathcal{L}$ to quadratic order. [6]

[Unseen] The kinetic energy takes the form

$$T = \partial_\mu \phi \partial^\mu \phi^* = \partial_\mu \sigma \partial^\mu \sigma + (v + \sigma)^2 \partial_\mu \theta \partial^\mu \theta \simeq \partial_\mu \sigma \partial^\mu \sigma + v^2 \partial_\mu \theta \partial^\mu \theta,$$

which we note is not canonically normalized. The potential takes the form

$$V = m^2(v + \sigma)^2 + 4\epsilon(v + \sigma)^2 \cos^2\theta + \lambda(v + \sigma)^4$$

Expanding to quadratic order we get

$$V = (m^2 + 4\epsilon)\sigma^2 + 4\epsilon v^2(1 - \theta^2/2)^2 + \lambda v^2 + 6\lambda v^2 \sigma^2 = (m^2 + 6\lambda v^2 + 4\epsilon)\sigma^2 - 4\epsilon v^2 \theta^2.$$

Taking into account the non-canonically normalized kinetic terms, we read off from the terms quadratic in the fields that the mass-squareds are given by

$$\left(m^2 + 6\lambda \frac{-m^2 - 4\epsilon}{2\lambda} + 4\epsilon\right) = (m^2 + 3(-m^2 - 4\epsilon) + 4\epsilon) = -2m^2 - 8\epsilon$$

(where in the second step we plugged in the value of v^2 found earlier) and

$$-4\epsilon,$$

so the masses may be written as

$$|2m^2 + 8\epsilon|^{1/2}$$

and

$$|4\epsilon|^{1/2}.$$

(g) By considering $|\epsilon| \ll |m^2|, v^2$, comment on your results in view of Goldstone's theorem. [2]

[Unseen] For $\epsilon = 0$ there would be an exact continuous symmetry and an associated Goldstone boson, according to Goldstone's theorem. If ϵ is non-zero but small, the corresponding particle acquires a small mass.

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3 (a) Briefly explain the concept of natural units. [2]

[Bookwork] Units such as $\hbar$ and c are merely conversion factors between historically distinct concepts of mass, length, time, etc . It is convenient to work in a system of units in which $\hbar = c = 1$, leaving only a single dimensionful scale, which we can take to be a mass. So energy has mass dimension one, time has mass dimension minus one, etc. In particular physics, it's usual to take the single dimensionful scale to be an energy, measured in GeV.

The dynamics of a real vector field A^μ in $2+1$ spacetime dimensions with co-ordinates x^ν is described, in natural units, by the lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda.$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\epsilon^{\mu\nu\lambda}$ is a totally antisymmetric tensor with $\epsilon^{012} = 1$, g is a real constant, and all greek indices take values in $\{0, 1, 2\}$.

(b) Find the mass dimension of the constant g. [2]

[Unseen] The lagrangian density is the sum of two terms, each of which must have the same mass dimension. Both terms contain the same power of the field A^μ, but the second term has one fewer derivative. So $[g] = [\partial_\mu]$. But $[\partial_\mu] = -[p/\hbar] = -[m/c\hbar]$, so g has dimensions of mass.

(c) Explain the concept of a gauge transformation and discuss whether or not the action obtained from $\mathcal{L}$ is gauge invariant. [5]

[Mostly seen] A gauge transformation of the field A_μ is defined to be $A_\mu \to A_\mu + \partial_\mu f$. We then have that $F_{\mu\nu}$ is gauge invariant, so we need only examine the second term in $\mathcal{L}$. This shifts by

$$\delta\mathcal{L} = \epsilon^{\mu\nu\lambda}\left(\partial_\mu f \partial_\nu A_\lambda + A_\mu \partial_\nu \partial_\lambda f\right).$$

Here the second term vanishes, because we have $\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda = 0$. The first term may be written as $\partial_\mu\left[\epsilon^{\mu\nu\lambda} f \partial_\nu A_\lambda\right]$, since again when the derivative acts on the second term in the product we can use $\epsilon^{\mu\nu\lambda}\partial_\mu\partial_\nu = 0$. Thus, the lagrangian density is not quite invariant, but rather shifts by a total derivative and so the action shifts by a contribution on the boundary. If the field A_μ and gauge parameter f vanish fast enough as $x^\mu \to \infty$, then we will indeed have gauge invariance.

(d) Starting from the Euler-Lagrange equations, derive the field equation

$$\partial_\mu F^{\mu\nu} + g\epsilon^{\nu\mu\rho}F_{\mu\rho} = 0.$$

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[4]

[Partly seen] Here, the derivation of the first term is standard. The Euler-Lagrange equations read

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} - \frac{\delta \mathcal{L}}{\delta A_\nu} = 0$$

and in $\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu}$ we get four contributions of $-\frac{1}{4}F^{\mu\nu}$ to get $-\partial_\mu F^{\mu\nu}$ *in toto*. The second term has two contributions. For $\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu}$, we get $g\epsilon^{\rho\mu\nu}A_\rho$, giving a piece $g\epsilon^{\rho\mu\nu}\partial_\mu A_\rho = g\epsilon^{\nu\rho\mu}\partial_\mu A_\rho$. For $\frac{\delta \mathcal{L}}{\delta A_\nu}$ we get $g\epsilon^{\nu\mu\lambda}\partial_\mu A_\lambda$. In total, we get $g\epsilon^{\nu\rho\mu}\partial_\mu A_\rho - g\epsilon^{\nu\rho\mu}\partial_\rho A_\mu = -g\epsilon^{\nu\rho\mu}F_{\mu\rho}$. Multiplying by -1, we get the desired field equation

$$\partial_\mu F^{\mu\nu} + g\epsilon^{\nu\mu\rho}F_{\mu\rho} = 0.$$

(e) Show that the quantity $\tilde{F}^\mu = \epsilon^{\mu\nu\rho}F_{\nu\rho}$ obeys the identity $\partial_\mu \tilde{F}^\mu = 0$. [2]

[Unseen] We have $\partial_\mu \tilde{F}^\mu = \epsilon^{\mu\nu\rho}\partial_\mu\left[\partial_\nu A_\rho - \partial_\rho A_\nu\right]$. For both terms the antisymmetric nature of $\epsilon^{\mu\nu\rho}$ kills the symmetric pair of partial derivatives, viz. $\partial_\mu\partial_\nu$ or $\partial_\mu\partial_\rho$. Hence $\partial_\mu \tilde{F}^\mu = 0$.

(f) Using the identity $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\beta\delta^\rho_\alpha$, show that $\tilde{F}^\mu$ obeys the field equation

$$(\partial_\nu\partial^\nu + m^2)\tilde{F}^\mu = 0,$$

where m^2 is a function of g whose form you should determine. [7]

[Unseen] Starting from $\tilde{F}^\mu = \epsilon^{\mu\nu\rho}F_{\nu\rho}$ and using the identity $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\beta\delta^\rho_\alpha$, we get $F^{\mu\nu} = \epsilon^{\mu\nu\alpha}\tilde{F}_\alpha/2$. So plugging this into the field equation $\partial_\mu F^{\mu\nu} + g\epsilon^{\nu\mu\rho}F_{\mu\rho} = 0$, we get

$$\epsilon^{\mu\alpha\beta}\partial_\mu \tilde{F}_\beta + 2g\tilde{F}^\alpha = 0.$$

Acting on this with $\epsilon_\alpha{}^{\gamma\delta}\partial_\gamma$ yields, for the first term $-\partial^2\tilde{F}^\delta$, where we used the identity $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\beta\delta^\rho_\alpha$ again, along with $\partial_\mu\tilde{F}^\mu = 0$, and for the second term $-4g^2\tilde{F}^\delta$, where we used the relation $\epsilon^{\mu\alpha\beta}\partial_\mu\tilde{F}_\beta + 2g\tilde{F}^\alpha = 0$. So in total we get the desired field equation, with $m^2 = (2g)^2$.

(g) Discuss whether or not the field A^μ can propagate over long distances and discuss how many polarizations it has. [3]

[Unseen] We see that the field equation for $\tilde{F}^\mu$ takes the form of independent Klein-Gordon equations for each component. So each propagates like a massive scalar and will lead to a short range propagation, exponentially surpressed by the 'mass' $2g$.

There will however only be two degrees of freedom, because the 3 components of $\tilde{F}^\mu$ are subject to the relation $\partial_\mu \tilde{F}^\mu = 0$. But this is what we expect for a massive vector field in $2+1$ dimensions: going to the rest frame, there should be two spin degrees of freedom, rather than 3 as in $3+1$ dimensions.

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4 A crystalline ferromagnet has a magnetisation described by a vector with real components m_1 and m_2 and is symmetric under $m_1 \to -m_1$, $m_2 \to -m_2$, and $m_1 \leftrightarrow m_2$.

(a) Assuming that the fields do not vary throughout space, explain why the Landau-Ginsburg free energy may be taken to be

$$f = \alpha \left[\frac{1}{2} t (m_1^2 + m_2^2) + \frac{1}{4} (m_1^4 + m_2^4 + 2\lambda m_1^2 m_2^2) \right],$$

where $t = \frac{T - T_c}{T_c}$ is the reduced temperature and α and λ are real constants. [6]

[Mostly seen] At large scales, we can expand in powers of the fields (and derivatives, though these are neglected here). We should write the most general terms consistent with the symmetries (on the basis that thermal flucatuations will generate such terms even if they are not present microscopically), which up to quartic order in the fields are $m_1^2 + m_2^2$, $m_1^2 m_2^2$, and $m_1^4 + m_2^4$. *A priori* these can have arbitrary coefficients, but we are also free to rescale m_1, m_2 by a common amount, and thus choose the coefficients of two terms to be the same. It remains to fix the temperature dependence of the coefficients. To get a phase transition, we need to expand the first term to linear order in temperature (and T_c is the constant term), while for the other terms it suffices to expand to zeroth order.

(b) Assuming $\alpha > 0$, for what values of λ is the free energy bounded below? [3]

[Partly seen] At large values of the fields, the free energy is dominated by the quartic terms, namely

$$m_1^4 + m_2^4 + 2\lambda m_1^2 m_2^2.$$

We may write this in matrix form as

$$\begin{pmatrix} m_1^2 & m_2^2 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} m_1^2 \\ m_2^2 \end{pmatrix}.$$

The eigenvalues of the 2×2 matrix are are $1 \pm \lambda$, with corresponding eigenvectors given by $\boldsymbol{e}_\pm \propto \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$. Only the first of these is realisable, since the vector $\begin{pmatrix} m_1^2 & m_2^2 \end{pmatrix}$ always has non-negative entries. So we see that the free energy is bounded below provided that the corresponding eigenvalue $1 + \lambda$ is non-negative, implying $\lambda \geq -1$.

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(c) By minimizing the free energy, find the number of physically-distinct phases and characterise each of them in terms of the values of m_1 and m_2. [6]

[Partly seen] By differentiating w.r.t. m_1, we get the field equation

$$m_1 t + m_1^3 + \lambda m_1 m_2^2 = 0$$

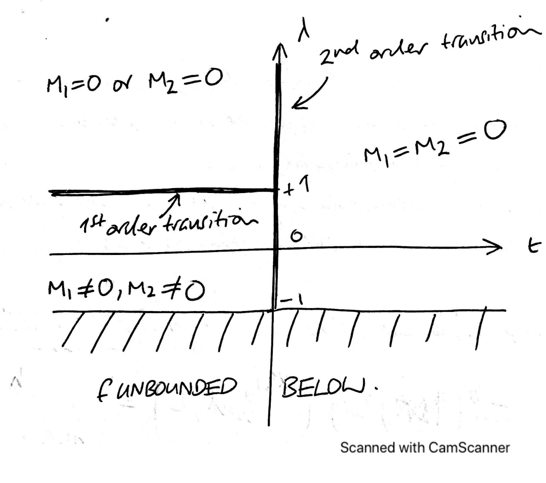
while the field equation for m_2 can be obtained by symmetry by swapping m_1 and m_2. We thus have *a priori* 4 possible cases:

1. $m_1 = 0$ and $m_2 = 0$, in which case we have $f = 0$;
2. $m_1 = 0$ and $m_2^2 = -t$, which clearly requires $t < 0$, such that $f = -\alpha \frac{t^2}{4}$;
3. $m_2 = 0$ and $m_1^2 = -t$, which because of the symmetry under interchange of m_1 and m_2 is physically indistinguishable and so is not a distinct phase.
4. $m_1^2 + \lambda m_2^2 = -t$ and $m_2^2 + \lambda m_1^2 = -t$, implying $m_1^2 = m_2^2 = \frac{-t}{1+\lambda}$, such that $f = -\alpha \frac{t^2}{2(1+\lambda)}$. Since $\lambda \geq -1$, we see that this requires $t < 0$.

We thus have three distinct phases, which may be characterised by having either both fields zero, one field zero, or neither field zero.

(d) Draw a phase diagram in the (t, λ) plane, taking care to indicate which phases occur where and the location and order of the phase transitions. [10]

[Partly seen]



For $t > 0$, there is only one possible phase, namely the one where both fields vanish. For $t < 0$ we have three possible phases, but the phases where at least one field is non-zero have negative free energy, so must be favoured over the $t > 0$ phase. Since

$$-\alpha \frac{t^2}{2(1+\lambda)} + \alpha \frac{t^2}{4} = t^2 \frac{\alpha}{4} \frac{\lambda - 1}{\lambda + 1},$$

we see that the third phase (with both fields non-vanishing is favoured when $\lambda < 1$. For the order of the phase transitions, it suffices to compute the magnetisation on either side of the transition. We find that the magnetisations are continuous at $t = 0$, but their derivatives are not, indicating a second order transition. In contrast, at $\lambda = 1$ we find that the magnetisations themselves are discontinuous, indicating a first order phase transition.

END OF PAPER