UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

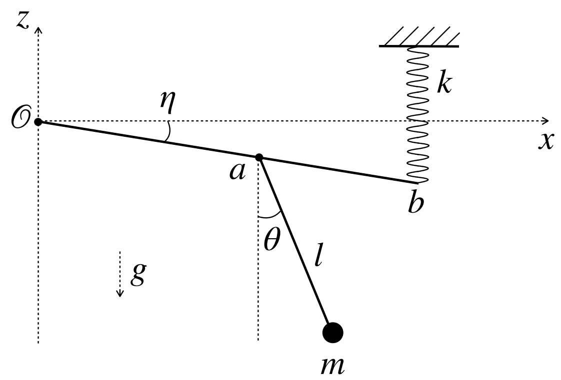
Friday 20 January 2023 2 pm to 4 pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains* **4** *sides, including this one.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 An end $\mathcal{O}$ of a massless rod of length b is fixed, allowing the rod to swing by an angle η in the vertical $x-z$ plane, as shown in the figure below. The other end of the rod is connected to a support by a spring of force constant k. The spring is sufficiently long that it can be considered to remain vertical at all times. At the natural spring length, the rod makes an angle η_0 with the x axis. The rod is connected, at a distance a from $\mathcal{O}$, to one end of a second massless rod of length l that is allowed to swing freely, making an angle θ with the vertical. A bob with a mass m is attached to the free end of the second rod.



(a) Show that the lagrangian of the system is given by

$$\begin{aligned} L &= \frac{1}{2}m\left[a^2\dot{\eta}^2 - 2al\sin(\eta+\theta)\dot{\eta}\dot{\theta} + l^2\dot{\theta}^2\right] \\ &+ mg(l\cos\theta + a\sin\eta) - \frac{1}{2}b^2k(\sin\eta - \sin\eta_0)^2. \end{aligned}$$

[6]

(b) Find the Euler-Lagrange equations of motion. [6]

(c) Assuming that the spring can freely intersect with the rods and the bob, find all equilibrium positions of the system. [5]

(d) Find the value of η_0 so that the first rod has an equilibrium position with $\eta = 0$ and find the values of the other constants for which such an equilibrium is possible. [3]

(e) Consider an equilibrium position with $\theta = \eta = 0$. By analysing small fluctuations, determine whether the equilibrium is stable or unstable. [4]

(f) Discuss the time-dependence of the lagrangian and its implications for the conserved quantities in this system. [1]

2 (a) State Noether's theorem for a complex scalar field ϕ whose dynamics is described by a lagrangian density $\mathcal{L}$ and derive the form of the conserved current J^μ. [4]

Consider the lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi\phi^* - \epsilon(\phi + \phi^*)^2 - \lambda(\phi\phi^*)^2.$$

(b) Assuming $\epsilon = 0$, find a continuous symmetry transformation that acts on ϕ but not on the spacetime coordinates and find the associated conserved current J^μ. [3]

(c) Now assume $\epsilon \neq 0$. Does the previously identified transformation remain a symmetry of the lagrangian density? If not, show how the lagrangian density transforms under an infinitesimal version of the same transformation. Are there any continuous or discrete symmetries that remain when $\epsilon \neq 0$? [3]

(d) By modifying your derivation in (a), find how the continuity equation $\partial_\mu J^\mu = 0$ changes when $\epsilon \neq 0$. [3]

For the remainder of this question, assume that $m^2 < 0$, $\epsilon < 0$, and $\lambda > 0$.

(e) Use the parameterisation $\phi = a + ib$, where a, b are real scalar fields, to find all possible values of ϕ where the system is in its ground state. [4]

(f) Use the parameterisation $\phi = (v + \sigma)e^{i\theta}$ to find the masses of the real scalar fields σ and θ in the ground state $\phi = v$ by expanding $\mathcal{L}$ to quadratic order. [6]

(g) By considering $|\epsilon| \ll |m^2|, v^2$, comment on your results in view of Goldstone's theorem. [2]

3 (a) Briefly explain the concept of natural units. [2]

The dynamics of a real vector field A^μ in $2+1$ spacetime dimensions with co-ordinates x^ν is described, in natural units, by the lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + g\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda.$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\epsilon^{\mu\nu\lambda}$ is a totally antisymmetric tensor with $\epsilon^{012} = 1$, g is a real constant, and all greek indices take values in $\{0, 1, 2\}$.

(b) Find the mass dimension of the constant g. [2]

(c) Explain the concept of a gauge transformation and discuss whether or not the action obtained from $\mathcal{L}$ is gauge invariant. [5]

(d) Starting from the Euler-Lagrange equations, derive the field equation

$$\partial_\mu F^{\mu\nu} + g\epsilon^{\nu\mu\rho} F_{\mu\rho} = 0.$$

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[4]

(e) Show that the quantity $\tilde{F}^\mu = \epsilon^{\mu\nu\rho} F_{\nu\rho}$ obeys the identity $\partial_\mu \tilde{F}^\mu = 0$. [2]

(f) Using the identity $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha \delta^\rho_\beta - \delta^\nu_\beta \delta^\rho_\alpha$, show that $\tilde{F}^\mu$ obeys the field equation

$$(\partial_\nu \partial^\nu + m^2)\tilde{F}^\mu = 0,$$

where m^2 is a function of g whose form you should determine. [7]

(g) Discuss whether or not the field A^μ can propagate over long distances and discuss how many polarizations it has. [3]

4 A crystalline ferromagnet has a magnetisation described by a vector with real components m_1 and m_2 and is symmetric under $m_1 \to -m_1$, $m_2 \to -m_2$, and $m_1 \leftrightarrow m_2$.

(a) Assuming that the fields do not vary throughout space, explain why the Landau-Ginsburg free energy may be taken to be

$$f = \alpha \left[\frac{1}{2} t(m_1^2 + m_2^2) + \frac{1}{4}(m_1^4 + m_2^4 + 2\lambda m_1^2 m_2^2) \right],$$

where $t = \frac{T - T_c}{T_c}$ is the reduced temperature and α and λ are real constants. [6]

(b) Assuming $\alpha > 0$, for what values of λ is the free energy bounded below? [3]

(c) By minimizing the free energy, find the number of physically-distinct phases and characterise each of them in terms of the values of m_1 and m_2. [6]

(d) Draw a phase diagram in the (t, λ) plane, taking care to indicate which phases occur where and the location and order of the phase transitions. [10]

END OF PAPER