NATURAL SCIENCES TRIPOS Part II

25th January 2022 10.30 am to 12.30 pm

# THEORETICAL PHYSICS I

*Attempt* **all 4** *questions. Answer each question in a separate booklet. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains 5 sides, including this one.*

1 A point mass of mass $m$ moves on a fixed circular ring of radius $a$ lying in a horizontal plane.

(a) **[Bookwork]** Show that the angular velocity of the point mass is constant. [3]

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In plane polar co-ordinates, the velocity is $a\dot{\theta}$, so the lagrangian reads [1/3]

$$L = \frac{1}{2} m a^2 \dot{\theta}^2$$

and the Euler-Langrange equation reads [2/3]

$$m a^2 \ddot{\theta} = 0,$$

implying $\dot{\theta}$ is constant.

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A second point mass of mass $M$ moves on a second fixed circular ring of radius $A$ lying in a horizontal plane a height $h$ above the first plane, such that the centres of the two rings are aligned vertically. The two point masses are attached to each other by a spring whose spring constant is $k$ and whose natural length vanishes.

(b) **[Unseen]** Write down the lagrangian of the system. [3]

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We use plane polar coordinates with angles $\theta$ and $\Theta$. The kinetic terms for the two point masses are as above and the potential energy comes from the stretched spring, whose squared extension is given by Pythagoras as [2/3]

$$(a\cos\theta - A\cos\Theta)^2 + (a\sin\theta - A\sin\Theta)^2 + h^2.$$

The lagrangian is therefore [1/3]

$$L = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} M A^2 \dot{\Theta}^2 - \frac{1}{2} k \left[ (a\cos\theta - A\cos\Theta)^2 + (a\sin\theta - A\sin\Theta)^2 \right],$$

where we have dropped an irrelevant constant.

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(c) **[Mostly seen]** Identify as many symmetries as you can and, where appropriate, find the corresponding conserved charges. [6]

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There is a discrete symmetry corresponding to interchange of the two rings, or equivalently to interchange of majuscules and miniscules. This has no corresponding conserved charge, but will provide a useful cross check of our results later on. [2/6]

The system is invariant under time translations, so the energy

$$\frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} M A^2 \dot{\Theta}^2 + \frac{1}{2} k \left[ (a\cos\theta - A\cos\Theta)^2 + (a\sin\theta - A\sin\Theta)^2 \right],$$

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is conserved. [2/6]

Finally, the system is invariant under a combined rotation of the angles $\theta$ and $\Theta$ (intuitively or see next part). The conserved quantity here is the total angular momentum: [2/6]

$$\frac{\partial L}{\partial \dot{\theta}} + \frac{\partial L}{\partial \dot{\Theta}} = ma^2\dot{\theta} + MA^2\dot{\Theta}\,.$$

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(d) **[Unseen]** Find all points of equilbrium of the system. [4]

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Given the symmetry under a combined rotation of the angles $\theta$ and $\Theta$, it is useful to define $\phi := \Theta - \theta$, in terms of which the potential takes the form

$$V = \frac{k}{2}\left[(a - A\cos\phi)^2 + A^2\sin^2\phi)\right] = -kAa\cos\phi,$$

where we again have dropped an irrelevant constant. [1/4]

This potential has a minimum at $\phi = 0$ and a maximum at $\phi = \pi$. Reassuringly, it is invariant under the exchange of majuscules and miniscules (which in turn sends $\phi$ to minus itself. [2/4]

It is important to note that these are not stationary minima or maxima because of the flat direction corresponding (in the new coordinates $\theta$ and $\phi$) to translations in $\theta$. [1/4]

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(e) **[Unseen]** For each equilibrium point, find the normal modes corresponding to small motions and discuss whether or not each point is stable. [6]

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For small oscillations about the 'minimum', we expand to get the potential $kaA\phi^2/2$ so the Euler-Langrange equations become [1/6]

$$\begin{aligned} ma^2\ddot{\theta} + MA^2(\ddot{\theta} + \ddot{\phi}) &= 0 \\ MA^2(\ddot{\theta} + \ddot{\phi}) &= -kaA\phi\,. \end{aligned}$$

To find the normal modes, we make an ansatz of the the form [1/6]

$$\begin{aligned} \theta &= \alpha e^{i\omega t} \\ \phi &= \beta e^{i\omega t}\,. \end{aligned}$$

Substitution yields [1/6]

$$\begin{aligned} -ma^2\alpha\omega^2 - MA^2\omega^2(\alpha + \beta) &= 0 \\ -MA^2\omega^2(\alpha + \beta) &= -kAa\beta\,. \end{aligned}$$

From the former, we see that either $\omega^2 = 0$, in which case the latter yields $\beta = 0$, or $\alpha + \beta = -\frac{ma^2}{MA^2}\alpha$, in which case the latter yields [2/6]

$$\omega^2 = k\frac{ma^2 + MA^2}{MmAa} = k\left[\frac{a}{MA} + \frac{A}{ma}\right]$$

.

For the 'maximum', the only effect after expansion is to change the sign of the potential, so we can recover all results for the normal modes by simply sending $k \mapsto -k$ in the previous expressions. [1/6]

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(f) [**Unseen**] Give reasons why your expressions for the normal modes are consistent with the symmetries of the system. [3]

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We have written the amplitude relation in terms of $\alpha + \beta$ and $\alpha$ because $\alpha + \beta$ is the amplitude for $\Theta$. We see that both the amplitudes and the normal frequencies are symmetric under interchange of majuscules and miniscules. [1/3]

Moreover, the appearance of the mode with $\omega^2 = 0$ is consistent with the symmetry under combined translation of the the two angular co-ordinates. [1/3]

Finally, one may show that the two charges (i.e., the total energy and the total angular momentum) are indeed conserved on these motions as follows. For, e.g., the stable mode, the angular momentum is just $ma^2\dot{\theta} + MA^2\dot{\Theta}$, which vanishes due to the amplitude relation $\alpha + \beta = -\frac{ma^2}{MA^2}\alpha$. For the energy, we just need to show that the amplitude of the kinetic part is equal to that of the potential part (since their phases differ by $\pi/2$). Ignoring an overall factor of $k/2$, the kinetic part goes as

$$\begin{aligned}\omega^2\left[ma^2\alpha^2 + MA^2(\alpha+\beta)^2\right] &= \alpha^2\left(\frac{a}{MA} + \frac{A}{ma}\right)\left(ma^2 + MA^2\frac{m^2a^4}{M^2A^4}\right) \\ &= Aa\left(1 + \frac{ma^2}{MA^2}\right)^2 .\end{aligned}$$

Meanwhile, the potential part goes as (again, ignoring an overall factor of $k/2$)

$$Aa\beta^2 = Aa\left(1 + \frac{ma^2}{MA^2}\right)^2 ,$$

as required. [1/3]

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2 A system in $1+1$ spacetime dimensions is described by a field $\phi(x,t)$ with a lagrangian density dependent on the field and its derivatives up to *second* order, $\mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi)$.

(a) [**Mostly Bookwork**] Derive the Euler-Lagrange equations for this system. [6]

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We have $\mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi)$, in a convenient covariant notation. The Euler-Lagrange equations are derived in the usual way, except that we must also consider the variation with respect to $\partial_\mu \partial_\nu \phi$. [1/6]

For these we need to integrate by parts twice, leading to a change of sign with respect to the terms involving first order derivatives. [1/6]

When the dust settles (calculations omitted in the model answer but expected in a solution), we get [4/6]

$$\partial_\mu \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \frac{\partial \mathcal{L}}{\partial \phi} = 0 \,.$$

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Transverse waves on a compressed rod are described by a $1+1$ spacetime dimensional real scalar field $\phi(x,t)$ with lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}\rho \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2}\alpha \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2}\beta \left( \frac{\partial^2 \phi}{\partial x^2} \right)^2 ,$$

where $\rho, \alpha, \beta > 0$.

(b) [**Bookwork**] Discuss qualitatively the physical origin of each of the terms in the lagrangian, including a justification of their signs. [5]

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The first term represents the kinetic energy of an element of the rod, so $\rho$ is the mass density per unit length, and should be positive, as usual. [1/5]

The second term represents the compressional force in the rod. Because the rod is under compression, rather than tension, this has the opposite sign to the usual one for transverse waves on a stretched string. It means that the rod is susceptible to buckling, since configurations in which the rod is curved rather than straight are energetically favoured. [2/5]

The third term represents a resistance to bending due to the internal rigidity of the beam. Here curvature leads to an increase in potential energy, so $\beta > 0$. [2/5]

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(c) [**Unseen**] Show that the action is invariant under shifts in $\phi$ and find the corresponding quantity that is conserved in time. [4]

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The lagrangian density features only derivatives of $\phi$, so the action is trivially invariant. [1/4]

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Using the usual derivation of Noether's theorem (one should take account of the higher derivative terms, but ultimately these do not affect the result), we have that the conserved charge is [3/4]

$$\int dx \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \int \rho \dot{\phi} dx \,.$$

(Calculations are omitted in the model answer but expected in a solution.)

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(d) **[Partly seen]** Using the Euler-Lagrange equations, show that the waves are dispersive. [4]

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Using the generic form of the Euler-Lagrange equations derived above, we get [2/4]

$$\frac{\partial}{\partial t}(\rho\dot{\phi}) + \frac{\partial}{\partial x}(\alpha\phi') + \frac{\partial^2}{\partial x^2}(\beta\phi'') = 0$$

or simply

$$\rho\ddot{\phi} + \alpha\phi'' + \beta\phi'''' = 0.$$

Considering a wave of the form $\phi = e^{i(kx-\omega t)}$, we find solutions when [1/4]

$$\omega^2 = \frac{k^2}{\rho}(\beta k^2 - \alpha).$$

From the dispersion relation, we easily see that the group velocity and phase velocity do not coincide, so the waves are dispersive. [1/4]

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Suppose now that such a rod of length $l$ has its ends fixed.

(e) **[Unseen]** By considering the dispersion relation, or otherwise, discuss whether the rod is stable or unstable. [6]

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Evidently, waves, and hence the beam itself, are unstable when $k^2 < \alpha/\beta$, because one mode is exponentially growing in time. [2/6]

When does a beam of length $l$ admit such a wave? The longest standing wave in such a beam has wavelength $2l$. So we get such a mode when [4/6]

$$k^2 = \left(\frac{2\pi}{2l}\right)^2 < \alpha/\beta \implies l > \pi\sqrt{\frac{\beta}{\alpha}}.$$

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3 (a) **[Bookwork]** Explain briefly the principles of mean field theory and under what conditions it is considered a reliable approximation. [3]

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Mean field theory approximates the interaction of a spin with its neighbours by assuming that its neighbours behave as a typical spin in the system. Namely, each spin interacts with the thermodynamic average of its neighbours. [2/3]

(Up to) an additional mark is given for either of the following:

- It becomes increasingly more reliable as the number of neighbours increases. [1/3]
- It is exact in the limit of infinite dimensions or infinite range interactions. [1/3]

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Consider spins $S_i$, each taking *three* possible values: $S_i = 0, \pm 1$, arranged on a square lattice with $N$ sites and periodic boundary conditions. The system has hamiltonian given by

$$H = -\frac{J}{2} \sum_{i,\delta} S_i S_{i+\delta} - B \sum_i S_i \,,$$

where $J > 0$ is the interaction coupling constant, $\delta$ labels the four neighbours of each site $i$, and $B$ is a magnetic field.

(b) **[Mostly bookwork]** Compute the free energy of the system in the mean field approximation, and express it in terms of $S$ (the order parameter, i.e., the mean field average value of a spin) and of $\beta = (k_B T)^{-1}$ (where $k_B$ is the Boltzmann constant and $T$ is the temperature). [4]

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In mean field theory, one assumes that each spin takes on a value close to the equilibrium thermodynamic average, $S$. Therefore, we write $S = (S_i - S) + S$ and expand the quadratic term in the energy to linear order in the 'small parameter' $S_i - S$:

$$S_i S_{i+\delta} = [(S_i - S) + S][(S_{i+\delta} - S) + S] \simeq (S_i - S)S + S(S_{i+\delta} - S) + S^2 = S_i S + S S_{i+\delta} - S^2$$

where the very last term is a constant and can be ignored. [1/4]

The mean field energy of the system can then be written as

$$\begin{aligned} E &= -\frac{J}{2} \sum_{i,\delta} (S_i S + S S_{i+\delta}) - B \sum_i S_i \\ &= -\frac{J}{2} \sum_{\delta} \left( \sum_i S_i + \sum_i S_{i+\delta} \right) S - B \sum_i S_i = -\left(4JS + B\right) \sum_i S_i \,, \end{aligned}$$

where in the last equality we used the fact that $\sum_i S_{i+\delta} = \sum_i S_i$ and that $\sum_\delta 1 = 4$ on the square lattice. [2/4]

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We have reduced the energy to a sum of one body terms, which allows for a straightforward calculation of the partition function:

$$\begin{aligned} Z &= \sum_{\{S_i\}} e^{-\beta E} = \left[ \sum_{S_i=0,\pm 1} e^{\beta(4JS+B)S_i} \right]^N \\ &= \{1 + 2\cosh\left[\beta\left(4JS + B\right)\right]\}^N , \end{aligned}$$

and from it one can obtain the free energy $f$: $\beta f = \ln Z$. [1/4]

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(c) [**Bookwork**] Explain what is meant by the mean field self-consistency condition and show that, for the problem at hand, it takes the form: [4]

$$S = \frac{2\sinh\left[\beta\left(4JS + B\right)\right]}{1 + 2\cosh\left[\beta\left(4JS + B\right)\right]} .$$

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Mean field theory assumes that each spin interacts with the average $S$ of its neighbours, which reduces the energy to one-body terms only, allowing to solve the thermodynamic problem exactly. One must then require the approximation to be internally consistent by computing the average value of a spin, $\langle S_i \rangle$, and setting it equal to $S$. [1/4]

This can be done by either using the definition $\langle S_i \rangle = (1/Z) \sum_{\{S_j\}} S_i e^{-\beta E}$, or via the thermodynamic relation $\langle S_i \rangle = (1/\beta) \partial/\partial B \ln Z$. A few lines of algebra produce the anticipated result: [3/4]

$$\langle S_i \rangle = \frac{2\sinh\left[\beta\left(4JS + B\right)\right]}{1 + 2\cosh\left[\beta\left(4JS + B\right)\right]} .$$

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(d) [**Bookwork but new algebra**] Consider the case $B = 0$. By means of a graphical method, or otherwise, demonstrate the existence of a critical point in the mean field approximation and find an expression for the critical temperature $T_c$ in terms of the parameters of the system. [3]

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Let us define for convenience $x = 4\beta JS$ and write the self-consistency condition as [1/3]

$$\frac{x}{4\beta J} = \frac{2\sinh x}{1 + 2\cosh x} .$$

The RHS (call it $F(x)$) tends to 1 for $x \to \pm\infty$. A few lines of algebra show that its first derivative

$$\frac{\partial F}{\partial x} = \frac{2(2 + \cosh x)}{(1 + 2\cosh x)^2}$$

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is always positive, and its second derivative

$$\frac{\partial^2 F}{\partial x^2} = -\frac{2(7 + 2\cosh x)\sinh x}{(1 + 2\cosh x)^3}$$

is positive for $x < 0$ and negative for $x > 0$. Therefore the shape of the function on the RHS is similar to $\tanh x$. [1/3]

Near $x = 0$, $F(x) \sim 2x/3$. If the slope of the LHS in the self-consistency equation is greater than $2/3$, there is only one solution at $x = 0$. If the slope is lesser than $2/3$, there are three solutions (and the stable ones are at finite $x$ – bonus mark if this is shown explicitly). The three solutions merge at $x = 0$ when the slope tends to $2/3$ from below. Therefore, the system undergoes a continuous phase transition, and the critical temperature is given by $2/3 = (4\beta J)^{-1}$, hence $T_c = 8J/(3k_B)$. [1/3]

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(e) **[Mostly bookwork]** Show that for temperatures slightly less than the critical temperature and for $B = 0$, the order parameter is given by [4]

$$S = \sqrt{\frac{8}{3}}\,|t|^{1/2} \qquad \text{where} \qquad t = \frac{T}{T_c} - 1\,.$$

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At temperatures $T < T_c$ near the critical point, $0 < x \ll 1$ and we can expand [1/4]

$$\begin{aligned} F(x) &= \frac{2x + x^3/3 + \mathcal{O}(x^5)}{3 + x^2 + \mathcal{O}(x^4)} = \left(\frac{2x}{3}\right)\frac{1 + x^2/6 + \mathcal{O}(x^4)}{1 + x^2/3 + \mathcal{O}(x^4)} \\ &= \frac{2x}{3}\left(1 - \frac{x^2}{6}\right) + \mathcal{O}(x^4)\,. \end{aligned}$$

We can then substitute this result into the self-consistency equation and remember the expression for the critical temperature found above, to arrive at the equation:

$$\frac{T_c}{T}\left(1 - \frac{x^2}{6}\right) \simeq 1\,,$$

where we ignored $\mathcal{O}(x^4)$ terms and simplified an overall factor $x$. [1/4]

This clearly gives the scaling exponent $\beta = 1/2$: $x \simeq \sqrt{6}\,(1 - T/T_c)^{1/2}$. [1/4]

Finally, to find the scaling form given in the question, we express the temperature in terms of $t$: $T = T_c(1 + t)$. And we use the relation: $x = 4\beta J S = 3T_c/(2T)S$. Substituting into the equation for $x$ above, we find

$$\frac{3T_c}{2T_c(1+t)}S \simeq \sqrt{6}\left(1 - \frac{T_c(1+t)}{T_c}\right)^{1/2} \qquad \to \qquad S \simeq \sqrt{\frac{8}{3}}\,|t|^{1/2}\,,$$

where we ignored higher order terms ($\sim |t|^{3/2}$). [1/4]

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(f) [**Bookwork**] Explain why the following equation is an appropriate Landau-Ginzburg free energy density $f$ for the microscopic problem at hand:

$$f = f_0(T) + a\,(T - T_c)\,m(x)^2 + \frac{1}{2}b\,m(x)^4 + c\left[\frac{dm(x)}{dx}\right]^2 - B\,m(x)$$

where $m(x)$ is a real scalar field and $a$, $b$, $c$ are real positive parameters. [3]

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Any of the following points will give one mark each, up to a total of 3 marks:

- scalar order parameter for the microscopic degrees of freedom coarse grains into a scalar field $m(x)$ [1/3]
- spatial field fluctuations are generally energetically costly, and they are suppressed by the leading order derivative term ($c > 0$) [1/3]
- the microscopic energy is symmetric in $\{S_i\} \to \{-S_i\}$, up to the Zeeman term, and so the coarse grained free energy must be symmetric in $m(x) \to -m(x)$ [1/3]
- $b > 0$ is a physical constraint to prevent the free energy from diverging (negatively) when the field $m(x)$ diverges [1/3]
- given $b > 0$, the phase transition is controlled by the coefficient of the quadratic term, which in this case was explicitly expanded to linear order in $T - T_c$ [1/3]

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(g) [**Unseen**] Consider the case where the term $B\,m(x)$ in the free energy above is replaced by $B^3\,m(x)$, and define the higher-order susceptibilities:

$$\chi^{(n)} = \left.\frac{\partial^n \overline{m}}{\partial B^n}\right|_{B=0}, \qquad n = 1, 2, 3, \dots\,,$$

where $\overline{m}$ is the equilibrium value of the order parameter in the saddle point approximation. Show that, for $T > T_c$, $\chi^{(1)} = \chi^{(2)} = 0$, and that $\chi^{(3)}$ diverges when $T$ approaches $T_c$ from above as $3/[a(T - T_c)]$. [4]

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In the saddle point approximation, the field is assumed to be uniform in space ($m(x) = \overline{m}$) and to minimise the free energy: [1/4]

$$\begin{aligned} f &= f_0 + a(T - T_c)\overline{m}^2 + \frac{1}{2}b\overline{m}^4 - B^3\overline{m} \\ \frac{\partial f}{\partial \overline{m}} &= 2a(T - T_c)\overline{m} + 2b\overline{m}^3 - B^3 = 0\,. \end{aligned}$$

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Instead of solving for $\overline{m}$ and computing $\chi^{(n)}$, it is more convenient to take derivatives of the equation that defines it: [1/4]

$$2a(T-T_c)\overline{m} + 2b\overline{m}^3 - B^3 = 0$$
$$2a(T-T_c)\frac{\partial \overline{m}}{\partial B} + 6b\overline{m}^2\frac{\partial \overline{m}}{\partial B} - 3B^2 = 0$$
$$2a(T-T_c)\frac{\partial^2 \overline{m}}{\partial B^2} + 12b\overline{m}\left(\frac{\partial \overline{m}}{\partial B}\right)^2 + 6b\overline{m}^2\frac{\partial^2 \overline{m}}{\partial B^2} - 6B = 0$$
$$2a(T-T_c)\frac{\partial^3 \overline{m}}{\partial B^3} + 12b\left(\frac{\partial \overline{m}}{\partial B}\right)^3 + 24b\overline{m}\frac{\partial \overline{m}}{\partial B}\frac{\partial^2 \overline{m}}{\partial B^2} + 12b\overline{m}\frac{\partial \overline{m}}{\partial B}\frac{\partial^2 \overline{m}}{\partial B^2} + 6b\overline{m}^2\frac{\partial^3 \overline{m}}{\partial B^3} - 6 = 0\,.$$

Evaluating them at $B=0$ as required, we find $\overline{m}=0$ (recall that the question sets $T > T_c$!) from the first equation. The second and third equations then give us $\chi^{(1)} = \chi^{(2)} = 0$. Finally, the fourth equation gives [2/4]

$$\chi^{(3)} = \frac{3}{a(T-T_c)}\,.$$

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4 Consider the following lagrangian density for 2 real scalar fields in 1+1 space-time dimensions, $\phi_1(x,t)$ and $\phi_2(x,t)$,

$$\mathcal{L} = \frac{\partial\phi_1}{\partial t}\frac{\partial\phi_2}{\partial t} - c^2\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x}\,,$$

where $c > 0$ is a real parameter with dimensions of velocity.

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Bonus mark if students comment that the energy of this system is not bounded from below. However, this is inconsequential to the questions asked in the paper.

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(a) **[Part bookwork, part new]** Find the field transformations of the form

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

that leave the lagrangian density invariant (where $\alpha$, $\beta$, $\gamma$, $\delta$ are real parameters). [4]

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A possible approach to answer the question goes as follows. The Lagrangian density can be conveniently written as:

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} \dot\phi_1 \\ \dot\phi_2 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot\phi_1 \\ \dot\phi_2 \end{pmatrix} - \frac{c^2}{2}\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix},$$

where dots and primes have the usual meaning. Transformations of the type given in the question leave the Lagrangian density invariant if and only if [1/4]

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 2\alpha\gamma & \alpha\delta+\beta\gamma \\ \alpha\delta+\beta\gamma & 2\beta\delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This leads to two possible choices: $\alpha = \delta = 0$, $\beta\gamma = 1$, or $\beta = \gamma = 0$, $\alpha\delta = 1$. And correspondingly two families of transformations: [3/4]

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \beta \\ 1/\beta & 0 \end{pmatrix}.$$

Alternatively, one could proceed by first doing a change of variables which diagonalizes the quadratic form.

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(b) **[Part bookwork, part new]** Recall that Noether's theorem applies only to transformations that are continuously connected to the identity. Compute the corresponding Noether current $J^\mu$ in the system and use the Euler-Lagrange equations to show that it satisfies the appropriate continuity equation, $\partial_\mu J^\mu = 0$. [7]

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Only the first family of transformations is continuously connected to the identity (for $\alpha \to 1$), and therefore one cannot apply Noether's theorem to the second family. [1/7]

In order to use Noether's theorem, we expand the first transformation to leading order near the identity, $\alpha = 1 + \varepsilon$ with $|\epsilon| \ll 1$: [2/7]

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \simeq \mathbb{I} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \varepsilon \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix} ,$$

and we find $\delta\phi_1 = \phi_1$, $\delta\phi_2 = -\phi_2$.

Using the definition of Noether's current in the lecture notes

$$J^\mu = \sum_{j=1}^{2} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_j} \delta\phi_j \, ,$$

we obtain $J^0 = \phi_1 \dot{\phi}_2 - \dot{\phi}_1 \phi_2$ and $J^1 = -c^2 \left( \phi_1 {\phi'}_2 - {\phi'}_1 \phi_2 \right)$. [2/7]

The Euler-Lagrange equations for this Lagrangian density are

$$\frac{\partial \mathcal{L}}{\partial \phi_j} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_j} = 0 \quad j = 1,2 \qquad \to \qquad \ddot{\phi}_j - c^2 {\phi''}_j = 0 \, ,$$

which straightforwardly imply [2/7]

$$\partial_\mu J^\mu = \frac{\partial J^0}{\partial t} + \frac{\partial J^1}{\partial x} = \phi_1 \left( \ddot{\phi}_2 - c^2 {\phi''}_2 \right) - \left( \ddot{\phi}_1 - c^2 {\phi''}_1 \right) \phi_2 = 0 \, .$$

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(c) [**Mostly bookwork**] Compute the stress energy tensor and show explicitly that it is conserved. If needed, you may assume the metric $g^{\mu\nu} = \mathrm{diag}(1, -1)$. [6]

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As explained in the lectures, one can conveniently compute $T^\mu_\nu$ without the need for a metric tensor. In the model answer below we show instead how to compute $T^{\mu\nu}$. Bonus mark to students who discuss accurately the relevance of the metric in this part of the question.

From the definition

$$T^{\mu\nu} = \sum_j \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_j} \partial^\nu \phi_j - g^{\mu\nu} \mathcal{L} \, ,$$

one finds $T^{00} = T^{11} = \dot{\phi}_1 \dot{\phi}_2 + c^2 {\phi'}_1 {\phi'}_2$, $T^{10} = c^2 T^{01} = -c^2 \left( \dot{\phi}_1 {\phi'}_2 + {\phi'}_1 \dot{\phi}_2 \right)$. (Recall that, with the given choice of metric, $\partial_0 = \partial^0 = \partial/\partial t$ and $\partial_1 = -\partial^1 = \partial/\partial x$.) [3/6]

In order to show that the stress energy tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$ (namely $\partial_0 T^{00} + \partial_1 T^{10} = 0$ and $\partial_0 T^{01} + \partial_1 T^{11} = 0$), we use the Euler-Lagrange equations obtained earlier, $\ddot{\phi}_j = c^2 \phi''_j$: [3/6]

$$\partial_0 \left[ \dot{\phi}_1 \dot{\phi}_2 + c^2 \phi'_1 \phi'_2 \right] - c^2 \partial_1 \left[ \dot{\phi}_1 \phi'_2 + \phi'_1 \dot{\phi}_2 \right] =$$
$$c^2 \left[ \phi''_1 \dot{\phi}_2 + \dot{\phi}_1 \phi''_2 + \dot{\phi}'_1 \phi'_2 + \phi'_1 \dot{\phi}'_2 - \dot{\phi}'_1 \phi'_2 - \dot{\phi}_1 \phi''_2 - \phi''_1 \dot{\phi}_2 - \phi'_1 \dot{\phi}'_2 \right] = 0$$
$$-\partial_0 \left[ \dot{\phi}_1 \phi'_2 + \phi'_1 \dot{\phi}_2 \right] + \partial_1 \left[ \dot{\phi}_1 \dot{\phi}_2 + c^2 \phi'_1 \phi'_2 \right] =$$
$$-c^2 \phi''_1 \phi'_2 - \dot{\phi}_1 \dot{\phi}'_2 - \dot{\phi}'_1 \dot{\phi}_2 - c^2 \phi'_1 \phi''_2 + \dot{\phi}'_1 \dot{\phi}_2 + \dot{\phi}_1 \dot{\phi}'_2 + c^2 \phi''_1 \phi'_2 + c^2 \phi'_1 \phi''_2 = 0 \, .$$

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(d) **[Unseen]** Now consider the modified lagrangian density

$$\mathcal{L} = \frac{\partial \phi_1}{\partial t} \frac{\partial \phi_2}{\partial t} - c^2 \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} + \frac{1}{2\tau} \left( \phi_1 \frac{\partial \phi_2}{\partial t} - \phi_2 \frac{\partial \phi_1}{\partial t} \right) ,$$

where $\tau > 0$ is a real parameter with dimensions of time. Solve the Euler-Lagrange equations for this lagrangian density, find the corresponding dispersion relations, and write the solutions in complex exponential form $\phi_j = A_j e^{i(k_j x - \omega_j t)}$ for $j = 1, 2$. [6]

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The new Euler-Lagrange equations take the form: [2/6]

$$\begin{aligned} -\frac{1}{\tau} \dot{\phi}_1 - \ddot{\phi}_1 + c^2 \phi''_1 &= 0 \\ +\frac{1}{\tau} \dot{\phi}_2 - \ddot{\phi}_2 + c^2 \phi''_2 &= 0 \, . \end{aligned}$$

The equations for the two fields are trivially decoupled and can be solved separately. [1/6]

Using complex exponential solutions of the form $\phi_1 = A_1 e^{i(k_1 x - \omega_1 t)}$ and $\phi_2 = A_2 e^{i(k_2 x - \omega_2 t)}$, we find the dispersion relations [2/6]

$$\begin{aligned} \omega_1^2 + \frac{i}{\tau} \omega_1 - c^2 k_1^2 &= 0 \qquad \to \qquad \omega_1 = -\frac{i}{2\tau} \pm \sqrt{c^2 k_1^2 - \frac{1}{4\tau^2}} \\ \omega_2^2 - \frac{i}{\tau} \omega_2 - c^2 k_2^2 &= 0 \qquad \to \qquad \omega_2 = \frac{i}{2\tau} \pm \sqrt{c^2 k_2^2 - \frac{1}{4\tau^2}} \, . \end{aligned}$$

Finally, we substitute into the solutions expressed in exponential form to find:

$$\begin{aligned} \phi_1 &= A_1 e^{i\left[k_1 x \mp \omega_1^{(0)} t\right]} e^{-t/(2\tau)} \\ \phi_2 &= A_2 e^{i\left[k_2 x \mp \omega_2^{(0)} t\right]} e^{t/(2\tau)} \, , \end{aligned}$$

where we defined for convenience of notation $\omega_j^{(0)} = \sqrt{c^2 k_j^2 - 1/(4\tau^2)}$. [1/6]

Bonus mark to students who discuss in detail physical and unphysical solutions (exponentially growing and decaying in time).

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(e) **[Unseen]** Discuss how the behaviour of the solutions changes for $c^2k_j^2 > 1/(4\tau^2)$ and for $c^2k_j^2 < 1/(4\tau^2)$. [2]

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For $c^2k_j^2 > 1/(4\tau^2)$, the argument of the square root is positive and $\omega_j^{(0)}$ is real. The corresponding solution has a propagating wave component, multiplied by an exponentially increasing / decreasing time-dependent factor, $e^{\mp t/(2\tau)}$. [1/2]

For $c^2k_j^2 < 1/(4\tau^2)$, the argument of the square root is negative and $\omega_j^{(0)}$ is purely imaginary. The corresponding solution has a spatially oscillatory component (standing wave), multiplied by an exponentially increasing / decreasing time-dependent factor, $e^{\pm|\omega_1^{(0)}|t-t/(2\tau)}$ and $e^{\pm|\omega_2^{(0)}|t+t/(2\tau)}$. [1/2]

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END OF PAPER