NATURAL SCIENCES TRIPOS Part II

25th January 2022 10.30 am to 12.30 pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. Answer each question in a separate booklet. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains 5 sides, including this one.*

1 A point mass of mass m moves on a fixed circular ring of radius a lying in a horizontal plane.

(a) [**Bookwork**] Show that the angular velocity of the point mass is constant. [3]

A second point mass of mass M moves on a second fixed circular ring of radius A lying in a horizontal plane a height h above the first plane, such that the centres of the two rings are aligned vertically. The two point masses are attached to each other by a spring whose spring constant is k and whose natural length vanishes.

(b) [**Unseen**] Write down the lagrangian of the system. [3]

(c) [**Mostly seen**] Identify as many symmetries as you can and, where appropriate, find the corresponding conserved charges. [6]

(d) [**Unseen**] Find all points of equilbrium of the system. [4]

(e) [**Unseen**] For each equilibrium point, find the normal modes corresponding to small motions and discuss whether or not each point is stable. [6]

(f) [**Unseen**] Give reasons why your expressions for the normal modes are consistent with the symmetries of the system. [3]

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2 A system in $1+1$ spacetime dimensions is described by a field $\phi(x,t)$ with a lagrangian density dependent on the field and its derivatives up to *second* order, $\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi)$.

(a) [**Mostly Bookwork**] Derive the Euler-Lagrange equations for this system. [6]

Transverse waves on a compressed rod are described by a $1+1$ spacetime dimensional real scalar field $\phi(x,t)$ with lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\alpha\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\beta\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 ,$$

where $\rho, \alpha, \beta > 0$.

(b) [**Bookwork**] Discuss qualitatively the physical origin of each of the terms in the lagrangian, including a justification of their signs. [5]

(c) [**Unseen**] Show that the action is invariant under shifts in ϕ and find the corresponding quantity that is conserved in time. [4]

(d) [**Partly seen**] Using the Euler-Lagrange equations, show that the waves are dispersive. [4]

Suppose now that such a rod of length l has its ends fixed.

(e) [**Unseen**] By considering the dispersion relation, or otherwise, discuss whether the rod is stable or unstable. [6]

3 (a) [**Bookwork**] Explain briefly the principles of mean field theory and under what conditions it is considered a reliable approximation. [3]

Consider spins S_i, each taking *three* possible values: $S_i = 0, \pm 1$, arranged on a square lattice with N sites and periodic boundary conditions. The system has hamiltonian given by

$$H = -\frac{J}{2} \sum_{i,\delta} S_i S_{i+\delta} - B \sum_i S_i \,,$$

where $J > 0$ is the interaction coupling constant, δ labels the four neighbours of each site i, and B is a magnetic field.

(b) [**Mostly bookwork**] Compute the free energy of the system in the mean field approximation, and express it in terms of S (the order parameter, i.e., the mean field average value of a spin) and of $\beta = (k_B T)^{-1}$ (where k_B is the Boltzmann constant and T is the temperature). [4]

(c) [**Bookwork**] Explain what is meant by the mean field self-consistency condition and show that, for the problem at hand, it takes the form: [4]

$$S = \frac{2 \sinh \left[\beta \left(4JS + B\right)\right]}{1 + 2 \cosh \left[\beta \left(4JS + B\right)\right]} \,.$$

(d) [**Bookwork but new algebra**] Consider the case $B = 0$. By means of a graphical method, or otherwise, demonstrate the existence of a critical point in the mean field approximation and find an expression for the critical temperature T_c in terms of the parameters of the system. [3]

(e) [**Mostly bookwork**] Show that for temperatures slightly less than the critical temperature and for $B = 0$, the order parameter is given by [4]

$$S = \sqrt{\frac{8}{3}}\, |t|^{1/2} \qquad \text{where} \qquad t = \frac{T}{T_c} - 1 \,.$$

(f) [**Bookwork**] Explain why the following equation is an appropriate Landau-Ginzburg free energy density f for the microscopic problem at hand:

$$f = f_0(T) + a\,(T - T_c)\, m(x)^2 + \frac{1}{2} b\, m(x)^4 + c \left[\frac{dm(x)}{dx}\right]^2 - B\, m(x)$$

where $m(x)$ is a real scalar field and a, b, c are real positive parameters. [3]

(g) [**Unseen**] Consider the case where the term $B\,m(x)$ in the free energy above is replaced by $B^3\, m(x)$, and define the higher-order susceptibilities:

$$\chi^{(n)} = \left.\frac{\partial^n \overline{m}}{\partial B^n}\right|_{B=0} \,, \qquad n = 1, 2, 3, \dots \,,$$

where $\overline{m}$ is the equilibrium value of the order parameter in the saddle point approximation. Show that, for $T > T_c$, $\chi^{(1)} = \chi^{(2)} = 0$, and that $\chi^{(3)}$ diverges when T approaches T_c from above as $3/[a(T - T_c)]$. [4]

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4 Consider the following lagrangian density for 2 real scalar fields in 1+1 space-time dimensions, $\phi_1(x,t)$ and $\phi_2(x,t)$,

$$\mathcal{L} = \frac{\partial \phi_1}{\partial t}\frac{\partial \phi_2}{\partial t} - c^2 \frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} \, ,$$

where $c > 0$ is a real parameter with dimensions of velocity.

(a) **[Part bookwork, part new]** Find the field transformations of the form

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

that leave the lagrangian density invariant (where α, β, γ, δ are real parameters). [4]

(b) **[Part bookwork, part new]** Recall that Noether's theorem applies only to transformations that are continuously connected to the identity. Compute the corresponding Noether current J^μ in the system and use the Euler-Lagrange equations to show that it satisfies the appropriate continuity equation, $\partial_\mu J^\mu = 0$. [7]

(c) **[Mostly bookwork]** Compute the stress energy tensor and show explicitly that it is conserved. If needed, you may assume the metric $g^{\mu\nu} = \mathrm{diag}(1, -1)$. [6]

(d) **[Unseen]** Now consider the modified lagrangian density

$$\mathcal{L} = \frac{\partial \phi_1}{\partial t}\frac{\partial \phi_2}{\partial t} - c^2 \frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} + \frac{1}{2\tau}\left(\phi_1 \frac{\partial \phi_2}{\partial t} - \phi_2 \frac{\partial \phi_1}{\partial t} \right) \, ,$$

where $\tau > 0$ is a real parameter with dimensions of time. Solve the Euler-Lagrange equations for this lagrangian density, find the corresponding dispersion relations, and write the solutions in complex exponential form $\phi_j = A_j e^{i(k_j x - \omega_j t)}$ for $j = 1, 2$. [6]

(e) **[Unseen]** Discuss how the behaviour of the solutions changes for $c^2 k_j^2 > 1/(4\tau^2)$ and for $c^2 k_j^2 < 1/(4\tau^2)$. [2]

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