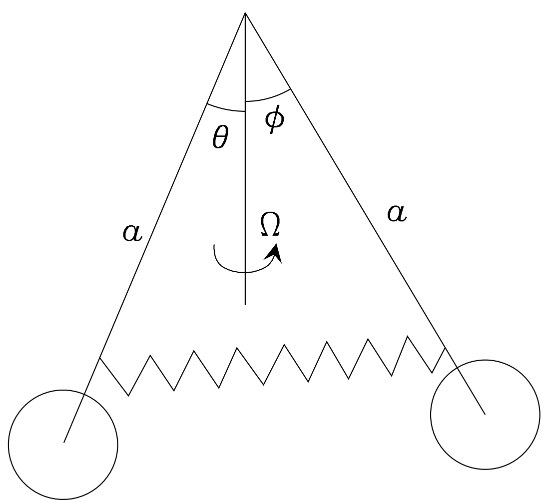
NATURAL SCIENCES TRIPOS Part II

Friday 22 January 2021 2pm to 4pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains ?? sides.*

1 The mechanical system shown in the Figure below consists of two bobs, each of mass m, each attached by a light rod of length a to a pivot rotating with constant angular velocity Ω about the vertical axis. The bobs are attached to each other by a spring whose spring constant is k and whose natural length vanishes.



(a) [**Bookwork**]Show that the sum of the gravitational potential energy and the energy stored in the spring may be written (up to a constant) as

$$-mga(\cos\theta + \cos\phi) - ka^2\cos(\theta+\phi),$$

where θ and ϕ are the angles between the rods and the downward vertical axis. [4]

For the gravitational potential energy, we evidently have the sum $-mga(\cos\theta + \cos\phi)$ relative to zero when the rods are horizontal and the minus sign reflects that the energy increases as the angle does. For the spring, we have that the length of the base is given by the cosine rule as $\frac{1}{2}k(a^2 + a^2 - 2a^2\cos(\theta+\phi)$; dropping the constant term gives us the desired result.

(b) [**Bookwork**]Find the lagrangian of the system. [3]

To the potential energy terms already derived we must add the kinetic energy terms. These contain a piece given by e.g. $\frac{1}{2}ma^2\dot{\theta}^2$ from the rotation of the rod in the plane of the rods, together with a piece $\frac{1}{2}m(a\sin\theta)^2\Omega^2$ coming from the rotation about the vertical axis. Putting everything together, we get

$$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2) + \frac{1}{2}ma^2\Omega^2(\sin^2\theta + \sin^2\phi) + mga(\cos\theta + \cos\phi) + ka^2\cos(\theta+\phi)$$

(c) [**Mostly seen**] Show that the system is invariant under time translations, and find the corresponding conserved quantity. What other symmetries does the system possess? [5]

The only place t appears in the lagrangian is in the derivatives, so the lagrangian is evidently time-translation-invariant; the corresponding conserved quantity may either be derived using Noether's theorem, or we may note that it is simply $T+V$, though we must take care that V also includes the terms without derivatives coming from the rotation about the vertical axis. Either way, we get that

$$E = \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2) - \frac{1}{2}ma^2\Omega^2(\sin^2\theta + \sin^2\phi) - mga(\cos\theta + \cos\phi) - ka^2\cos(\theta+\phi).$$

The system is also symmetric under the discrete $\mathbb{Z}/2$ symmetry given by $\phi \leftrightarrow \theta$. There is no conserved current, but this will nevertheless be useful in what follows. The system is also symmetric under rotations about the vertical axis.

(d) [**Unseen**] Show that, when $\Omega^2 \neq g/a$ and the angles θ and ϕ are small but non-vanishing, the only equilibrium positions occur at $\theta = \phi$. [4]

We derive the equilibrium positions by differentiating V with respect to the 2 angles. We obtain

$$0 = ma^2\Omega^2\sin\theta\cos\theta - mga\sin\theta - ka^2\sin(\theta+\phi),$$

together with (by symmetry) the same equation with $\phi \leftrightarrow \theta$. Subtracting one from the other, we get

$$0 = ma^2\Omega^2(\sin\theta\cos\theta - \sin\phi\cos\phi) - mga(\sin\theta - \sin\phi).$$

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When the angles are small enough and $\Omega^2 \neq g/a$, this reduces to a linear equation in the angles with just one solution, namely $\theta = \phi$. (For larger angles, there is always a solution with $\theta = \phi$, but there may be other solutions.)

(e) **[Unseen]** Find the equilibrium points with $\theta = \phi$, show that one normal frequency at such an equilibrium point with $\theta = \theta_0$ is given by

$$\frac{1}{2\pi}\sqrt{g/a\cos\theta_0 - \Omega^2\cos 2\theta_0},$$

and find the other normal frequency. [6]

When $\theta = \phi$, we find that either $\sin\theta = 0$ or $\cos\theta = \frac{g/a}{\Omega^2 - 2k/m}$ in equilibirum, using the equations above. The second derivatives of the lagrangian about such a point are

$$\frac{\partial^2 L}{\partial\theta^2} = (m\Omega^2 - k)a^2\cos 2\theta_0 - mga\cos\theta_0 \tag{1}$$

$$\frac{\partial^2 L}{\partial\phi^2} = (m\Omega^2 - k)a^2\cos 2\theta_0 - mga\cos\theta_0 \tag{2}$$

$$\frac{\partial^2 L}{\partial\theta\partial\phi} = -ka^2\cos 2\theta_0 \tag{3}$$

So the Euler-Lagrange equations for the normal modes yield

$$\begin{pmatrix}\ddot\theta \\ \ddot\phi\end{pmatrix} = \begin{pmatrix} x & y \\ y & x\end{pmatrix}\begin{pmatrix}\theta \\ \phi\end{pmatrix}$$

with $x := (\Omega^2 - k/m)\cos 2\theta_0 - g/a\cos\theta_0, y := -k/m\cos 2\theta_0$.

The normal frequencies are found from the eigenvalues $\omega^2 = -x \pm y = g/a\cos\theta_0 - \Omega^2\cos 2\theta_0, g/a\cos\theta_0 - \Omega^2\cos 2\theta_0 + 2k/m\cos 2\theta_0$. Taking the square roots and dividing by 2π gives the normal frequencies.

(f) **[Unseen]** Give a sufficient condition for such an equilibrium point to be stable, in terms of θ_0 and the other parameters. [3]

As Ω increases, the normal frequencies become imaginary, indicating that the equilibrium point is unstable. The system moves to an equilibrium point at larger angles. To find the critical value of Ω, we seek the point at which the lowest eigenvalue vanishes. This is $0 = g/a\cos\theta_0 - \Omega^2\cos 2\theta_0$, so we need $\Omega < \sqrt{\frac{g\cos\theta_0}{a\cos 2\theta_0}}$.

2 A fluid moving in 2+1 dimensional spacetime with co-ordinates x^μ, with $\mu \in \{0,1,2\}$, is described by 2 real fields $\varphi^i(x^\mu)$, with $i \in \{1,2\}$, and has lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{det}A$$

where A is the 2×2 matrix whose ijth element is $A^{ij} = \partial^\mu \varphi^i \partial_\mu \varphi^j$.

(a) [**Bookwork**] Show that $\mathrm{det}A = \partial^\mu\varphi^1\partial_\mu\varphi^1\partial^\nu\varphi^2\partial_\nu\varphi^2 - \partial^\mu\varphi^1\partial_\mu\varphi^2\partial^\nu\varphi^1\partial_\nu\varphi^2$ [3]

This follows immediately from the definition of the determinant of a 2×2 matrix. We use distinct pairs of dummy indices μ and ν so that the summation convention can be used.

(b) [**Unseen**] Show that for small oscillations about the equilibrium point $\varphi^i = x^i$, such that $\varphi^i = x^i + \pi^i$, the lagrangian density may be approximated by

$$\mathcal{L} = \frac{1}{2}\partial_0\pi^i\partial_0\pi^i - \frac{1}{2}(\partial_i\pi^i)^2.$$

[8]

We expand $\varphi^i = x^i + \pi^i$, keeping terms up to quadratic order in the fields. So for the first term in the lagrangian density we need the product of

$$\begin{aligned} \partial^\mu\varphi^1\partial_\mu\varphi^1 &= (\partial_0\varphi^1)^2 - (\partial_1\varphi^1)^2 - (\partial_2\varphi^1)^2 && (4)\\ &= (\dot{\pi}^1)^2 - (1+\partial_1\pi^1)^2 - (\partial_2\pi^1)^2 && (5)\\ &= -1 - 2\partial_1\pi^1 + (\dot{\pi}^1)^2 - (\partial_1\pi^1)^2 - (\partial_2\pi^1)^2 && (6) \end{aligned}$$

and

$$\begin{aligned} \partial^\mu\varphi^2\partial_\mu\varphi^2 &= (\dot{\pi}^2)^2 - (\partial_1\pi^2)^2 - (1+\partial_2\pi^2)^2 && (7)\\ &= -1 - 2\partial_2\pi^2 + (\dot{\pi}^2)^2 - (\partial_2\pi^2)^2 - (\partial_1\pi^2)^2 && (8) \end{aligned}$$

while for the second term we need

$$-\partial^\mu\varphi^1\partial_\mu\varphi^2\partial^\nu\varphi^1\partial_\nu\varphi^2 = -\left[\dot{\pi}^1\dot{\pi}^2 - (1+\partial_1\pi^1)\partial_1\pi^2 - (1+\partial_2\pi^2)\partial_2\pi^1\right]^2 \simeq \left[\partial_1\pi^2 + \partial_2\pi^1\right]^2.$$

In total, the terms up to quadratic order are

$$\mathcal{L} = -\frac{1}{2}\left[1 + 2(\partial_1\pi^1 + \partial_2\pi^2) - (\dot{\pi}^1)^2 - (\dot{\pi}^2)^2 + (\partial_1\pi^1 + \partial_2\pi^2)^2 + 2(\partial_1\pi^1\partial_2\pi^2 - \partial_1\pi^2\partial_2\pi^1)\right]$$

But the first, second, and final terms are all total derivatives, so we write this more simply as

$$\mathcal{L} = +\frac{1}{2}\dot{\pi}^i\dot{\pi}^i - \frac{1}{2}(\partial_i\pi^i)^2$$

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as required.

(c) [**Mostly seen**] By Fourier expanding $\pi^i = \int d^3k^\mu a^i(k^\mu) e^{ik_\mu x^\mu}$, calculate the dispersion relations $k^0(k^i)$ for longitudinal and transverse waves and give an explanation in terms of the physics of fluids. [5]

Plugging the given Fourier expansion into either the equations of motion or directly into the lagrangian density, we see that the models have dispersion relations satisfying

$$0 = (k^0)^2(a^i)^2 - (k^i a^i)^2.$$

So if we choose polarisation vector $a^i \propto k^i$ parallel to the spatial wavevector, we find a dispersion relation given by $\omega^2 = k^2$, which correspond to sound waves (with a sound speed equal to 1 in these units). Whereas if we choose polarisation vector a^i perpendicular to the spatial wavevector, we find a dispersion relation given by $\omega^2 = 0$. This appears odd at first glance, but closer inspection shows it to be not so. We know that fluids possess vortex excitations, which infinitesimally are indeed transverse. Such vortices can have any spatial extent, and moreover they can have arbitrarily small energy no matter what their spatial extent. So we must have $\omega = 0$.

An equivalent explanation is that fluids have no shear modulus, so shear waves have zero frequency.

(d) [**Unseen**] Suppose the lagrangian density is replaced by the more general expression

$$\mathcal{L}_f = -\frac{1}{2} f(\det A),$$

where f is an arbitrary function. Find an expression for the speed of sound in the fluid in terms of the derivatives of f. [5]

For the more general lagrangian density, we can essentially ring the changes. But we must take care in that when we perform the expansion in small fluctuations, we get contributions from higher derivatives of f. Indeed we have

$$\mathcal{L} = -\frac{1}{2}\left[f(1) + f'(1)(\det A - 1) + \frac{1}{2} f''(1)(\det A - 1)^2 + \dots \right] \tag{9}$$

$$= -\frac{1}{2}\left[f'(1)(-(\dot{\pi}^i)^2 + (\partial_i \pi^i)^2) + \frac{1}{2} f''(1)(2\partial_i \pi^i)^2 \right] \tag{10}$$

where we have ignored various total derivative and terms which are more than quadratic in the fields. We thus now have

$$\mathcal{L} = \frac{1}{2} f'(1)(\dot{\pi}^i)^2 - \frac{1}{2}(f'(1) + 2f''(1))(\partial_i \pi^i)^2.$$

Thus, comparing with the dispersion relation for longitudinal modes derived in the last part, we read off that the sound speed is given by $c_s = \sqrt{1 + 2f''(1)/f'(1)}$.

(e) **[Partly unseen]** Identify as many symmetries as you can of the lagrangian $\mathcal{L}_f$. [4]

The lagrangian is obviously invariant under Lorentz transformations of x^μ, as well as spacetime translations, making up the Poincaré group. The fact that the lagrangian is derivatively coupled shows that translations in the target space fields ϕ^i are symmetries and the summation of indices also shows that we can rotate the components of ϕ^i. But in fact the group of internal symmetries is much larger, because the determinant is invariant under any change of coordinates of ϕ^i which do not change areas, of which translations and rotations are a tiny subset. In fact there are infinitely many conserved charges here, which can be shown to correspond directly to Kelvin's theorem on the conservation of circulation in a fluid.

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3 A system is described by the lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu \boldsymbol{N} \cdot \partial_\mu \boldsymbol{N},$$

where $\boldsymbol{N}(x^\mu) \in \mathbb{R}^3$ is a vector field.

(a) Show that $\boldsymbol{N} \to \tilde{\boldsymbol{N}} = \boldsymbol{N} + \boldsymbol{\phi} \times \boldsymbol{N}$, where $\boldsymbol{\phi} \in \mathbb{R}^3$ are the infinitesimal transformation parameters, is a symmetry transformation of the lagrangian and find the associated conserved charges. [6]

We have

$$\begin{aligned} \tilde{N}_i \tilde{N}_i &= (N_i + \epsilon_{ijk}\phi_j N_k)(N_i + \epsilon_{ijk}\phi_j N_k) && (11)\\ &= N_i N_i + 2\epsilon_{ijk} N_i N_k \phi_j && (12)\\ &= N_i N_i, && (13) \end{aligned}$$

by the antisymmetry of the Levi-Civita tensor with respect to transpositions and terms have been kept to first order in $\boldsymbol{\phi}$.

We have $\delta N_i = \epsilon_{ijk}\phi_j N_k$ for $i = 1, 2, 3$. The associated Noether current is given by

$$\begin{aligned} J^\mu &= \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu N_i)} \delta N_i \\ &= \partial^\mu N_i \epsilon_{ijk} \phi_j N_k. \end{aligned}$$

The corresponding conserved charge is

$$\begin{aligned} Q &= \int \mathrm{d}^3\boldsymbol{r}\; J^0 \\ &= \int \mathrm{d}^3\boldsymbol{r}\; \pi_i \cdot \epsilon_{ijk}\phi_j N_k \\ &= \phi_j \int \mathrm{d}^3\boldsymbol{r}\; \epsilon_{jki} N_k \pi_i. \end{aligned}$$

Because we can choose any $\boldsymbol{\phi} \in \mathbb{R}^3$ there are three independently conserved charges which make up the conserved angular momentum vector

$$\boldsymbol{L} = \int \mathrm{d}^3\boldsymbol{r}\; \boldsymbol{N} \times \boldsymbol{\pi}.$$

Consider now the space-time transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^{0\mu\alpha\beta}\theta_\alpha x_\beta,$$

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where $\epsilon^{\nu\mu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor and the infinitesimal transformation parameters are described by the real four-vector $\theta^\mu = (0, \theta^1, \theta^2, \theta^3)$. All expressions will be given to first order in θ^μ and you should only work to this order.

(b) [**Unseen**] Show that

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\mu} - \epsilon^{0\mu\alpha\beta}\theta_\alpha \frac{\partial}{\partial x'^\beta}.$$

[3]

$$\begin{aligned}
\frac{\partial}{\partial x^\mu} x^\sigma &= \left(\frac{\partial}{\partial x'^\mu} - \epsilon^{0\mu\alpha\beta}\theta_\alpha \frac{\partial}{\partial x'^\beta}\right)\left(x'^\sigma - \epsilon^{0\sigma\alpha\beta}\theta_\alpha x'_\beta\right) \\
&= \delta^\sigma_\mu - \left(\epsilon^{0\sigma\alpha\mu} + \epsilon^{0\mu\alpha\sigma}\right)\theta_\alpha \\
&= \delta^\sigma_\mu
\end{aligned}$$

as required.

(c) [**Mostly seen**] Hence, show that the field transformation $\boldsymbol{N}(x^\mu) \to \tilde{\boldsymbol{N}}(x^\mu) = \boldsymbol{N}(x'^\mu)$ changes the action only by a boundary term. Show that the conserved charges associated with this symmetry transformation are given by

$$Q^\sigma = \epsilon^{0\sigma\alpha\beta} \int \mathrm{d}^3\boldsymbol{r}\; x_\alpha\, \partial_0 \boldsymbol{N} \cdot \partial_\beta \boldsymbol{N}.$$

[10]

$$\begin{aligned}
\sum_i \frac{\partial \tilde{N}_i}{\partial x^\mu}\frac{\partial \tilde{N}_i}{\partial x_\mu} &= \sum_i \left(\frac{\partial}{\partial x'^\mu} - \epsilon^{0\mu\alpha\beta}\theta_\alpha \frac{\partial}{\partial x'^\beta}\right)\tilde{N}_i \left(\frac{\partial}{\partial x'_\mu} - \epsilon^{0}{}_{\mu}{}^{\alpha\beta}\theta_\alpha \frac{\partial}{\partial x'^\beta}\right)\tilde{N}_i && (14)\\
&= \sum_i \left(\frac{\partial \tilde{N}_i}{\partial x'^\mu}\frac{\partial \tilde{N}_i}{\partial x'_\mu} - \frac{\partial \tilde{N}_i}{\partial x'_\mu}\frac{\partial \tilde{N}_i}{\partial x'^\beta}\epsilon^{0\mu\alpha\beta}\theta_\alpha - \frac{\partial \tilde{N}_i}{\partial x'_\mu}\frac{\partial \tilde{N}_i}{\partial x'^\beta}\epsilon^{0\mu\alpha\beta}\theta_\alpha\right) && (15)\\
&= \sum_i \frac{\partial \tilde{N}_i}{\partial x'^\mu}\frac{\partial \tilde{N}_i}{\partial x'_\mu} && (16)
\end{aligned}$$

because $\frac{\partial \tilde{N}_i}{\partial x'_\mu}\frac{\partial \tilde{N}_i}{\partial x'^\beta}$ is symmetric for $\mu, \beta \neq 0$. Hence,

$$\mathcal{L}\left(\tilde{N}_i, \frac{\partial \tilde{N}_i}{\partial x^\mu}\right) = \mathcal{L}\left(N_i(x'^\mu), \frac{\partial N_i(x'^\mu)}{\partial x'^\mu}\right) = \mathcal{L}(x'^\mu). \qquad (17)$$

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To find the associated conserved charges, express

$$\delta\mathcal{L} = \sum_i \left(\frac{\partial\mathcal{L}}{\partial N_i}\delta N_i + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu N_i\right)}\partial_\mu \delta N_i \right) \tag{18}$$

$$= \sum_i \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu N_i\right)}\delta N_i \right] = \epsilon^{0\mu\alpha\beta}\theta_\alpha x_\beta \partial_\mu \mathcal{L}. \tag{19}$$

Hence,

$$\partial_\mu \left[\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu N_i\right)}\delta N_i - \epsilon^{0\mu\alpha\beta}\theta_\alpha x_\beta \mathcal{L} \right] = 0 \tag{20}$$

With $\delta N_i = \epsilon^{0\mu\alpha\beta}\theta_\alpha x_\beta \partial_\mu N_i$, this gives the associated conserved charge as

$$Q = \sum_i \int \mathrm{d}^3\boldsymbol{r}\ \partial_0 N_i \cdot \epsilon^{0\mu\alpha\beta}\theta_\alpha x_\beta \partial_\mu N_i \tag{21}$$

$$= \theta_\alpha \sum_i \int \mathrm{d}^3\boldsymbol{r}\ \partial_0 N_i \cdot \epsilon^{0\alpha\beta\mu} x_\beta \partial_\mu N_i \tag{22}$$

Considering different choices of θ_α gives the three independently conserved charges given in the question.

(d) Deduce the reduced rotation symmetry when the term $(\nabla\cdot\boldsymbol{N})^2$ is added to $\mathcal{L}$, find the associated conserved charges, and interpret their physical meaning. [6]

In the presence of the term $(\nabla\cdot\boldsymbol{N})^2$ in $\mathcal{L}_\sigma$, the lagrangian is no longer symmetric under rotation of $\boldsymbol{r}$ or rotation of $\boldsymbol{N}$. However, it is symmetric under a *simultaneous* rotation of $\boldsymbol{N}$ and $\boldsymbol{r}$ in opposite directions and by the same angle. Hence, it is now the *difference* of intrinsic and orbital angular momenta which is conserved

$$\boldsymbol{J}_{\mathrm{tot}} = \int \mathrm{d}^3\boldsymbol{r}\ \left(\boldsymbol{N}\times\boldsymbol{\pi} - \sum_i \boldsymbol{x}\times\pi_i \boldsymbol{\nabla} N_i \right). \tag{23}$$

This first term corresponds to intrinsic angular momentum, because there is no dependence on $\boldsymbol{k}$ in the Fourier-domain, i.e. the angular momentum can be non-zero even if there are only Fourier modes with $\boldsymbol{k}=\boldsymbol{0}$.

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4 The lagrangian density of an electromagnetic field interacting with charged matter is given by

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this question we will look at charge-neutral systems with a bound current, such that $j_0 = A_0 = 0$.

(a) By parametrising the bound current as $\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{M} + \partial_0 \boldsymbol{P}$, where $\boldsymbol{M}, \boldsymbol{P} \in \mathbb{R}^3$ are the magnetisation and polarisation fields respectively, show that for a charge-neutral system the action can be rewritten in terms of the lagrangian density

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \boldsymbol{M} \cdot \boldsymbol{B} + \boldsymbol{P} \cdot \boldsymbol{E},$$

where the magnetic and electric fields $\boldsymbol{B}$ and $\boldsymbol{E}$ are defined with respect to A^μ in the usual way. [5]

We have

$$\begin{aligned}
-j_\mu A^\mu &= \boldsymbol{j} \cdot \boldsymbol{A} & (24)\\
&= (\boldsymbol{\nabla} \times \boldsymbol{M} + \partial_t \boldsymbol{P}) \cdot \boldsymbol{A} & (25)\\
&= \epsilon_{ijk} A_i \partial_j M_k + A_i \partial_0 P_i & (26)\\
&= \epsilon_{ijk} \partial_j (A_i M_k) - \epsilon_{ijk} M_k \partial_j A_i + \partial_0 (A_i P_i) - P_i \partial_0 A_i & (27)\\
&= \epsilon_{kji} M_k \partial_j A_i - P_i \partial_0 A_i & (28)\\
&= \boldsymbol{M} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{P} \cdot \partial_0 \boldsymbol{A} & (29)\\
&= \boldsymbol{M} \cdot \boldsymbol{B} + \boldsymbol{P} \cdot \boldsymbol{E}, & (30)
\end{aligned}$$

where boundary terms that do not affect the action have been neglected.

(b) A ferromagnet is described by the lagrangian density

$$\mathcal{L}_{\text{FM}} = \mathcal{L}_{\text{em}} - \frac{t}{2} \boldsymbol{M}^2 - u \left(\boldsymbol{M}^2\right)^2,$$

with $\boldsymbol{P} = \boldsymbol{0}$ and $u > 0$, t constant. A magnetic field of strength $h > 0$ is applied along the positive x^1-direction. By considering the appropriate Euler-Lagrange equation, write down the equation satisfied by $\boldsymbol{M}$. Hence, find the zero-field susceptibility $(\partial \boldsymbol{M} / \partial h)_{h \to 0^+}$. [7]

The Euler-Lagrange equation for $\boldsymbol{M}$ is given by

$$\begin{aligned}
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu M_i)} \right) &= \frac{\partial \mathcal{L}}{\partial M_i} & (31)\\
0 &= -t M_i - 4u |\boldsymbol{M}|^2 M_i + B_i & (32)
\end{aligned}$$

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For $\boldsymbol{B} = h\hat{\boldsymbol{x}} \neq \mathbf{0}$, we only have magnetisation along x_1. Differentiating the equation for $i = 1$ with respect to h, we obtain

$$0 \;=\; -t\chi - 12uM_1^2\chi + 1, \tag{33}$$

where $\chi = (\partial M_1/\partial h)$. The magnetisation as $\boldsymbol{B} = h\hat{\boldsymbol{x}} \to \mathbf{0}$ is given by

$$M_1^2 = \begin{cases} 0 & t > 0, \\ \frac{-t}{4u} & t < 0. \end{cases}$$

Hence, the zero-field susceptibility is given by

$$\chi = \begin{cases} \frac{1}{t} & t > 0, \\ \frac{1}{-2t} & t < 0. \end{cases}$$

(c) Now consider instead a plasma, described by the lagrangian density

$$\mathcal{L}_{\mathrm{d}} = \mathcal{L}_{\mathrm{em}} + \frac{1}{2m^2}\left(\partial_0 \boldsymbol{P}\right)^2,$$

with $\boldsymbol{M} = \mathbf{0}$ and m a constant. By considering the appropriate Euler-Lagrange equation and ensuring that causality is respected, show that the bound current in the plasma is given by

$$\boldsymbol{j}(\omega) = m^2 \lim_{\epsilon \to 0^+} \frac{\boldsymbol{E}(\omega)}{i\omega + \epsilon},$$

where Fourier transforms with respect to time are defined in the usual way, e.g. $\boldsymbol{j}(\omega) = \int \mathrm{d}t\, \boldsymbol{j}(t) e^{-i\omega t}$. You can assume that all of the current in the plasma is generated in response to a non-zero electric field. [6]

The Euler-Lagrange equation for $\boldsymbol{P}$ is given by

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu P_i\right)} \right) \;=\; \frac{\partial \mathcal{L}}{\partial P_i} \tag{34}$$

$$\frac{1}{m^2}\partial_0^2 P_i \;=\; E_i \tag{35}$$

$$\frac{1}{m^2}\partial_0 j_i \;=\; E_i, \tag{36}$$

where the last line follows from the fact that for $\boldsymbol{M} = \mathbf{0}$, the bound current is given by $j_i = \partial_0 P_i$. The Green's function of the above equation is given by

$$G(t) = m^2 \Theta(t), \tag{37}$$

where $\Theta(t)$ is the Heaviside step function and respects causality as $G(t < 0) = 0$. Fourier transforming the Heaviside step function we obtain

$$G(\omega) = \lim_{\epsilon \to 0^+} \frac{m^2}{i\omega + \epsilon} \tag{38}$$

and $j_i(\omega) = G(\omega)E_i(\omega)$, assuming all current is generated in response to the electric field (i.e. vanishing complementary function). Let's check that causality is respected. $G(\omega)$ has a pole at $\omega = i\epsilon$ with a residue of $-im^2$ in the upper half-plane. Evaluating

$$G(t) = m^2 \oint_C \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{i\omega + \epsilon} = \begin{cases} m^2 & t > 0, \\ 0 & t < 0, \end{cases}$$

where we have to close the contour C in the upper half-plane for $t > 0$ and in the lower half-plane for $t < 0$.

(d) Working in the Lorenz gauge $\partial_\mu A^\mu = 0$, write down the Euler-Lagrange equation satisfied by A^μ in the plasma and find the dispersion relation of the field. Compare the behaviour of the gauge field in the plasma with the Higgs mechanism. [7]

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) &= \frac{\partial \mathcal{L}}{\partial A_\nu} && (39) \\ -\frac{1}{2}\partial_\mu (F^{\mu\nu} - F^{\nu\mu}) &= j^\nu && (40) \\ \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu &= j^\nu && (41) \\ \partial_\mu \partial^\mu A^\nu &= j^\nu && (42) \end{aligned}$$

in the Lorentz gauge. Fourier transforming the equation, we obtain

$$-k_\mu k^\mu A^\nu(k^\sigma) = j^\nu(k^\sigma). \tag{43}$$

For a charge-neutral system, this reduces to

$$(\omega^2 - \boldsymbol{k}^2)\boldsymbol{A}(k^\sigma) = -\boldsymbol{j}(k^\sigma). \tag{44}$$

Substituting the expression for the current in a dielectric $\boldsymbol{j}(\omega) = \frac{-i\omega m^2 \boldsymbol{A}(\omega)}{i\omega + \epsilon}$, we obtain

$$(\omega^2 - \boldsymbol{k}^2)\boldsymbol{A}(k^\sigma) = m^2 \frac{i\omega \boldsymbol{A}(k^\sigma)}{i\omega + \epsilon}. \tag{45}$$

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For $\omega \neq 0$, we obtain the following dispersion

$$\omega^2 = \boldsymbol{k}^2 + m^2. \tag{46}$$

Dynamical (propagating, $\omega \neq 0$) Fourier modes of $\boldsymbol{A}(k^\sigma)$ are massive, as in the Higgs mechanism. However, static ($\omega = 0$) Fourier modes satisfy the massless Laplace's equation

$$\boldsymbol{k}^2 \boldsymbol{A}(\omega = 0, \boldsymbol{k}) = 0. \tag{47}$$

A static magnetic field is completely unaffected by the plasma as no currents are generated. This is unlike the Higgs mechanism, where we have

$$\left(\boldsymbol{k}^2 + m^2\right) \boldsymbol{A}(\omega = 0, \boldsymbol{k}) = 0, \tag{48}$$

or in real space

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = m^2 \boldsymbol{A}(\boldsymbol{r}), \tag{49}$$

which, for example, leads to the expulsion of the magnetic field from a superconductor (Meissner effect).

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