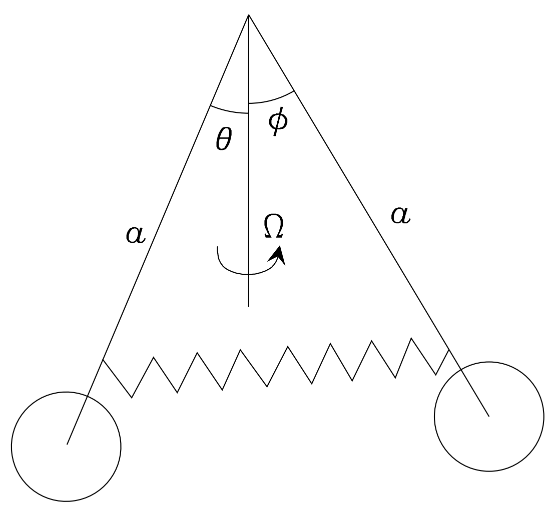
NATURAL SCIENCES TRIPOS Part II

Friday 22 January 2021 2pm to 4pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains 4 sides.*

1 The mechanical system shown in the Figure below consists of two bobs, each of mass m, each attached by a light rod of length a to a pivot rotating with constant angular velocity Ω about the vertical axis. The bobs are attached to each other by a spring whose spring constant is k and whose natural length vanishes.



(a) Show that the sum of the gravitational potential energy and the energy stored in the spring may be written (up to a constant) as

$$-mga(\cos\theta + \cos\phi) - ka^2\cos(\theta+\phi),$$

where θ and ϕ are the angles between the rods and the downward vertical axis. [4]

(b) Find the lagrangian of the system. [3]

(c) Show that the system is invariant under time translations, and find the corresponding conserved quantity. What other symmetries does the system possess? [5]

(d) Show that, when $\Omega^2 \neq g/a$ and the angles θ and ϕ are small but non-vanishing, the only equilibrium positions occur at $\theta = \phi$. [4]

(e) Find the equilibrium points with $\theta = \phi$, show that one normal frequency at such an equilibrium point with $\theta = \theta_0$ is given by

$$\frac{1}{2\pi}\sqrt{g/a\cos\theta_0 - \Omega^2\cos 2\theta_0},$$

and find the other normal frequency. [6]

(f) Give a sufficient condition for such an equilibrium point to be stable, in terms of θ_0 and the other parameters. [3]

2 A fluid moving in 2+1 dimensional spacetime with co-ordinates x^μ, with $\mu \in \{0,1,2\}$, is described by 2 real fields $\varphi^i(x^\mu)$, with $i \in \{1,2\}$, and has lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{det}A$$

where A is the 2×2 matrix whose ijth element is $A^{ij} = \partial^\mu\varphi^i\partial_\mu\varphi^j$.

(a) Show that $\mathrm{det}A = \partial^\mu\varphi^1\partial_\mu\varphi^1\partial^\nu\varphi^2\partial_\nu\varphi^2 - \partial^\mu\varphi^1\partial_\mu\varphi^2\partial^\nu\varphi^1\partial_\nu\varphi^2$ [3]

(b) Show that for small oscillations about the equilibrium point $\varphi^i = x^i$, such that $\varphi^i = x^i + \pi^i$, the lagrangian density may be approximated by

$$\mathcal{L} = \frac{1}{2}\partial_0\pi^i\partial_0\pi^i - \frac{1}{2}(\partial_i\pi^i)^2.$$

[8]

(c) By Fourier expanding $\pi^i = \int d^3k^\mu a^i(k^\mu)e^{ik_\mu x^\mu}$, calculate the dispersion relations $k^0(k^i)$ for longitudinal and transverse waves and give an explanation in terms of the physics of fluids. [5]

(d) Suppose the lagrangian density is replaced by the more general expression

$$\mathcal{L}_f = -\frac{1}{2}f(\mathrm{det}A),$$

where f is an arbitrary function. Find an expression for the speed of sound in the fluid in terms of the derivatives of f. [5]

(e) Identify as many symmetries of the lagrangian $\mathcal{L}_f$ as you can. [4]

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3 A system is described by the lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu \boldsymbol{N} \cdot \partial_\mu \boldsymbol{N},$$

where $\boldsymbol{N}(x^\mu) \in \mathbb{R}^3$ is a vector field.

(a) Show that $\boldsymbol{N} \to \tilde{\boldsymbol{N}} = \boldsymbol{N} + \boldsymbol{\phi} \times \boldsymbol{N}$, where $\boldsymbol{\phi} \in \mathbb{R}^3$ are the infinitesimal transformation parameters, is a symmetry transformation of the action and find the associated conserved charges. [6]

Consider now the space-time transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^{0\mu\alpha\beta}\theta_\alpha x_\beta,$$

where $\epsilon^{\nu\mu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor and the infinitesimal transformation parameters are described by the real four-vector $\theta^\mu = (0, \theta^1, \theta^2, \theta^3)$. All expressions will be given to first order in θ^μ and you should only work to this order.

(b) Show that

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\mu} - \epsilon^{0\mu\alpha\beta}\theta_\alpha \frac{\partial}{\partial x'^\beta}.$$

[3]

(c) Hence, show that the field transformation $\boldsymbol{N}(x^\mu) \to \tilde{\boldsymbol{N}}(x^\mu) = \boldsymbol{N}(x'^\mu)$ changes the action only by a boundary term. Show that the conserved charges associated with this symmetry transformation are given by

$$Q^\sigma = \epsilon^{0\sigma\alpha\beta} \int \mathrm{d}^3\boldsymbol{r}\; x_\alpha\, \partial_0 \boldsymbol{N} \cdot \partial_\beta \boldsymbol{N}.$$

[10]

(d) Deduce the reduced rotation symmetry when the term $(\nabla \cdot \boldsymbol{N})^2$ is added to $\mathcal{L}$, find the associated conserved charges, and interpret their physical meaning. [6]

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4 The lagrangian density of an electromagnetic field interacting with charged matter is given by

$$\mathcal{L}_{\text{em}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j_\mu A^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this question we will look at charge-neutral systems with a bound current, such that $j_0 = A_0 = 0$.

(a) By parametrising the bound current as $\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{M} + \partial_0 \boldsymbol{P}$, where $\boldsymbol{M}, \boldsymbol{P} \in \mathbb{R}^3$ are the magnetisation and polarisation fields respectively, show that for a charge-neutral system the action can be rewritten in terms of the lagrangian density

$$\mathcal{L}_{\text{em}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \boldsymbol{M} \cdot \boldsymbol{B} + \boldsymbol{P} \cdot \boldsymbol{E},$$

where the magnetic and electric fields $\boldsymbol{B}$ and $\boldsymbol{E}$ are defined with respect to A^μ in the usual way. [5]

(b) A ferromagnet is described by the lagrangian density

$$\mathcal{L}_{\text{FM}} = \mathcal{L}_{\text{em}} - \frac{t}{2}\boldsymbol{M}^2 - u\left(\boldsymbol{M}^2\right)^2,$$

with $\boldsymbol{P} = \boldsymbol{0}$ and $u > 0$, t constant. A magnetic field of strength $h > 0$ is applied along the positive x^1-direction. By considering the appropriate Euler-Lagrange equation, write down the equation satisfied by $\boldsymbol{M}$. Hence, find the zero-field susceptibility $(\partial \boldsymbol{M}/\partial h)_{h\to 0^+}$. [7]

(c) Now consider instead a plasma, described by the lagrangian density

$$\mathcal{L}_{\text{d}} = \mathcal{L}_{\text{em}} + \frac{1}{2m^2}\left(\partial_0 \boldsymbol{P}\right)^2,$$

with $\boldsymbol{M} = \boldsymbol{0}$ and m a constant. By considering the appropriate Euler-Lagrange equation and ensuring that causality is respected, show that the bound current in the plasma is given by

$$\boldsymbol{j}(\omega) = m^2 \lim_{\epsilon\to 0^+} \frac{\boldsymbol{E}(\omega)}{i\omega + \epsilon},$$

where Fourier transforms with respect to time are defined in the usual way, e.g. $\boldsymbol{j}(\omega) = \int \mathrm{d}t\, \boldsymbol{j}(t)e^{-i\omega t}$. You can assume that all of the current in the plasma is generated in response to a non-zero electric field. [6]

(d) Working in the Lorenz gauge $\partial_\mu A^\mu = 0$, write down the Euler-Lagrange equation satisfied by A^μ in the plasma and find the dispersion relation of the field. Compare the behaviour of the gauge field in the plasma with the Higgs mechanism. [7]

END OF PAPER