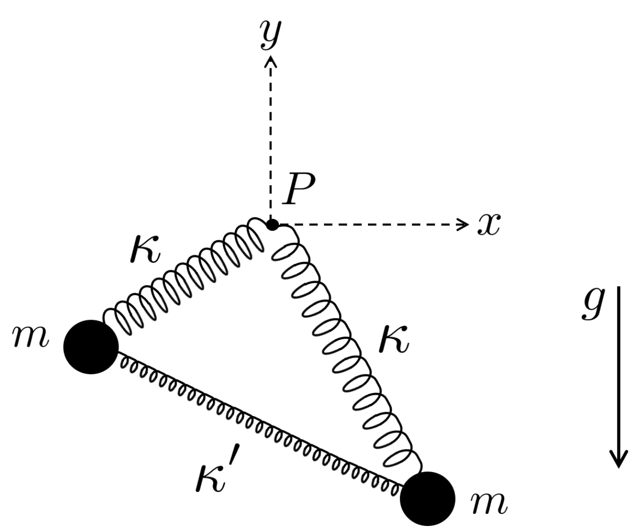
NATURAL SCIENCES TRIPOS Part II

Wednesday 17 January 2018

THEORETICAL PHYSICS I

Answers

1 Two identical beads of mass m are each attached to a pivot point P by a light spring of constant κ and unstretched length $\ell = 0$, in the presence of a gravitational acceleration g. They are further connected to one another by a spring of constant κ' and unstretched length $l' > 0$. The centres of the two beads are confined to move within a vertical plane through P, as sketched below.



(a) **[part bookwork, part unseen calculation]** Show that the Lagrangian of the system can be written as

$$\begin{aligned} L &= m\left(\dot{X}^2 + \dot{Y}^2\right) - 2mg\,(Y-\alpha) - \kappa\left[X^2 + (Y-\alpha)^2\right] \\ &\quad + m\left(\dot{x}^2 + \dot{y}^2\right) - \kappa\left(x^2+y^2\right) - 2\kappa'\left(\sqrt{x^2+y^2} - \frac{\ell'}{2}\right)^2 , \end{aligned}$$

where $x = (x_1 - x_2)/2$, $y = (y_1 - y_2)/2$, $X = (x_1 + x_2)/2$, $Y = (y_1 + y_2)/2 + \alpha$. Here, (x_1, y_1) and (x_2, y_2) denote the coordinates of the centres of the two beads in the Cartesian reference frame given by the dashed axes in the figure, and α is a generic constant. Find the Euler-Lagrange equations of motion of the system. [7]

The kinetic energy of the two beads can be written as [1]

$$T = \frac{1}{2} m \sum_i (\dot{x}_i^2 + \dot{y}_i^2) = m\left(\dot{x}^2 + \dot{y}^2 + \dot{X}^2 + \dot{Y}^2\right) .$$

And the potential energy is the sum of the gravitational energy of the beads and the elastic energy of three springs: [2]

$$\begin{aligned} V_g &= mg(y_1 + y_2) = 2mg\,(Y - \alpha) \\ V_e &= \frac{1}{2}\kappa \sum_i (x_i^2 + y_i^2) + \frac{1}{2}\kappa' \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} - \ell'\right)^2 \\ &= \kappa\left[x^2 + X^2 + y^2 + (Y - \alpha)^2\right] + 2\kappa' \left(\sqrt{x^2 + y^2} - \frac{\ell'}{2}\right)^2 . \end{aligned}$$

Recalling that the Lagrangian is then the difference between the kinetic and potential energies, $L = T - V_g - V_e$, we arrive at the Lagrangian given in the question.

The Euler-Lagrange equations [1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

where q is a system coordinate, for X, Y, x, and y, give the equations of motion: [3]

$$\begin{aligned} m\ddot{X} &= -\kappa X \\ m\ddot{Y} &= -\kappa Y - mg + \kappa\alpha \\ m\ddot{x} &= -\kappa x - 2\kappa' \left(\sqrt{x^2 + y^2} - \frac{\ell'}{2}\right) \frac{x}{\sqrt{x^2 + y^2}} \\ m\ddot{y} &= -\kappa y - 2\kappa' \left(\sqrt{x^2 + y^2} - \frac{\ell'}{2}\right) \frac{y}{\sqrt{x^2 + y^2}} . \end{aligned}$$

(b) **[bookwork]** Find the equilibrium positions of the beads and show that, for an appropriate choice of α, they satisfy [6]

$$X = Y = 0 , \qquad x^2 + y^2 = \left(\frac{\kappa'\ell'}{\kappa + 2\kappa'}\right)^2 .$$

In order to obtain the equilibrium positions, we set the time derivatives of the dynamical variables to zero in the equations of motion, and we find [1]

$$\begin{aligned} X &= 0\,, \\ Y + \frac{mg}{\kappa} - \alpha &= 0\,, \\ x\left[\kappa + 2\kappa'\left(\sqrt{x^2+y^2} - \ell'/2\right)\frac{1}{\sqrt{x^2+y^2}}\right] &= 0\,, \\ y\left[\kappa + 2\kappa'\left(\sqrt{x^2+y^2} - \ell'/2\right)\frac{1}{\sqrt{x^2+y^2}}\right] &= 0\,. \end{aligned}$$

The former two equations reduce to the form asked in the question for the choice of $\alpha = mg/\kappa$. [1]

The latter two are satisfied if $x = y = 0$, or if $\sqrt{x^2+y^2} = \kappa'\ell'/(\kappa+2\kappa')$. Notice, however, that $x = y = 0$ is in fact not acceptable: in order to derive the equations of motion, we assumed $x^2 + y^2 > 0$; indeed, if we set $x = y = 0$ in the original problem, the two beads are on top of one another and the forces acting on them do not vanish – which tells us clearly that this is not an equilibrium position. We are therefore left with $\sqrt{x^2+y^2} = \kappa'\ell'/(\kappa+2\kappa')$. The latter condition corresponds to the relative coordinate of one bead with respect to the other forming a circle of radius $\kappa'\ell'/(\kappa+2\kappa')$. [4]

Bonus mark if one comments about the potential crossing of the springs in the set of equilibrium solutions, and how to set conditions in order to avoid it.

(c) [**unseen calculation**] For the value of α chosen in (b), find the normal modes and the corresponding frequencies of small oscillations about the equilibrium positions. [8]

From the Euler-Lagrange equations found in (a) we have that the motion of X and Y is harmonic with frequency $\Omega = \sqrt{\kappa/m}$ around the equilibrium position $(0, 0)$. [2]

[Notice that the degeneracy of the frequency in both directions is due to the rotational symmetry in the (X, Y) plane.]

In order to find the small fluctuations about equilibrium for the x, y coordinates, we need to expand the equations of motion found in (a) around the equilibrium position $x = x_0 + \delta x$, $y = y_0 + \delta y$ up to linear order in small displacements δx and δy:

$$\begin{aligned} m\,\delta\ddot{x} &= -\frac{(\kappa+2\kappa')^3}{(\kappa' l')^2} x_0\left(x_0\delta x + y_0\delta y\right) + \mathcal{O}(\delta x^2, \delta y^2, \delta x\delta y), \\ m\,\delta\ddot{y} &= -\frac{(\kappa+2\kappa')^3}{(\kappa' l')^2} y_0\left(x_0\delta x + y_0\delta y\right) + \mathcal{O}(\delta x^2, \delta y^2, \delta x\delta y), \end{aligned}$$

where we used the fact that the equilibrium positions (x_0, y_0) fulfil $\sqrt{x_0^2 + y_0^2} = \kappa' l'/(\kappa+2\kappa')$ [see point (b)] to observe that the constant terms cancel out as expected at equilibrium, while the linear terms stem only from the expansion of $x_0\kappa' l'/\sqrt{x^2+y^2} - x_0\kappa' l'/\sqrt{x_0^2+y_0^2}$, as the linear terms in $-\kappa(x-x_0) - 2\kappa'(x-x_0) + (x-x_0)\kappa' l'/\sqrt{x_0^2+y_0^2}$ cancel out. [3]

Either by direct observation, or by diagonalisation of the matrix describing the linear differential equations

$$\begin{pmatrix} \ddot{\delta x} \\ \ddot{\delta y} \end{pmatrix} = -\frac{(\kappa+2\kappa')^3}{m(\kappa' l')^2} \begin{pmatrix} x_0^2 & x_0 y_0 \\ x_0 y_0 & y_0^2 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \tag{1}$$

we obtain a zero-frequency mode $(\delta x, \delta y) \propto (y_0, -x_0)$, and the other mode $(\delta x, \delta y) \propto (x_0, y_0)$ with frequency $\omega = \sqrt{(\kappa+2\kappa')/m}$. [3]

[The zero frequency is a result of the rotational symmetry in the (x, y) plane, as in the linear order this mode corresponds to the motion along the coordinates with the same energy and thus no force is experienced.]

(d) **[part unseen calculation, part bookwork]** Discuss the continuous symmetries of the Lagrangian and find the corresponding conserved quantities.

[Up to 3 bonus marks]. [4]

Bonus mark for noting that, since the Lagrangian is time-independent, the Hamiltonian is conserved. *[Notice, however, that only the action is invariant under time-translations, while that Lagrangian L itself is not invariant/symmetric, as $L(q(t), \dot{q}(t)) \mapsto L(q(t+\delta t), \dot{q}(t+\delta t))$ when $t \mapsto t+\delta t$. In contrast, when L is independent from a coordinate q, L is invariant under constant-in-time translations $q \mapsto q + \delta q$, and the corresponding linear momentum, $\partial L/\partial \dot{q}$, is conserved. For the system discussed here, however, Lagrangian depends on X, Y, x and y and no linear momenta are conserved (see also comment below).]*

We now discuss two continuous symmetries that leave the Lagrangian invariant. The marks are given for the discussion of one of them, and 2 bonus marks for discussion of both symmetries.

If we define the vector variables $\boldsymbol{\phi} = (x, y)^T$ and $\boldsymbol{\Phi} = (X, Y)^T$, then the Lagrangian can be rewritten as

$$L = m\left(|\dot{\boldsymbol{\phi}}|^2 + |\dot{\boldsymbol{\Phi}}|^2\right) + \frac{m^2 g^2}{\kappa} - \kappa\left(|\boldsymbol{\phi}|^2 + |\boldsymbol{\Phi}|^2\right) - 2\kappa'\left(|\boldsymbol{\phi}| - \frac{\ell'}{2}\right)^2,$$

which is clearly invariant under independent rotations $\boldsymbol{\phi} \to R\boldsymbol{\phi}$ and $\boldsymbol{\Phi} \to R'\boldsymbol{\Phi}$. [2]

In polar coordinates, $\boldsymbol{\phi} = (r, \theta)$ and $\boldsymbol{\Phi} = (R, \Theta)$, the Lagrangian

$$L = m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{R}^2 + R^2\dot{\Theta}^2\right) + \frac{m^2 g^2}{\kappa} - \kappa\left(r^2 + R^2\right) - 2\kappa'\left(r - \frac{\ell'}{2}\right)^2$$

does not depend on θ nor Θ, and the corresponding canonical momenta are therefore conserved, $p_\theta = mr^2\dot{\theta}$ and $p_\Theta = mR^2\dot{\Theta}$. In the original coordinates, we have that $p_\theta = m(x\dot{y} - y\dot{x})$ and $p_\Theta = m(X\dot{Y} - Y\dot{X})$. [2]

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[Notice that if we further express these quantities in terms of the coordinates and velocities of the two beads, we obtain:

$$\begin{aligned} x\dot{y} - y\dot{x} &= x_1\dot{y}_1 - \dot{x}_1 y_1 + x_2\dot{y}_2 - \dot{x}_2 y_2 \\ &\quad -x_1\dot{y}_2 + \dot{x}_1 y_2 - x_2\dot{y}_1 + \dot{x}_2 y_1 \\ X\dot{Y} - Y\dot{X} &= x_1\dot{y}_1 - \dot{x}_1 y_1 + x_2\dot{y}_2 - \dot{x}_2 y_2 \\ &\quad +x_1\dot{y}_2 - \dot{x}_1 y_2 + x_2\dot{y}_1 - \dot{x}_2 y_1 - \frac{2mg}{k}(\dot{x}_1 + \dot{x}_2)\,. \end{aligned}$$

Summing the two conserved quantities and dividing by 2*, one obtains*

$$x_1\dot{y}_1 - \dot{x}_1 y_1 + x_2\dot{y}_2 - \dot{x}_2 y_2 - \frac{mg}{k}(\dot{x}_1 + \dot{x}_2)$$

which is proportional to $J^z_{\mathrm{tot}} - (mg/\kappa)P^x_{\mathrm{tot}}$*, where* $J^z_{\mathrm{tot}} = m\sum_i(x_i\dot{y}_i - \dot{x}_i y_i)$ *is the* z *component of the total angular momentum and* $P^x_{\mathrm{tot}} = m(\dot{x}_1 + \dot{x}_2)$ *is the* x *component of the total linear momentum of the two beads.]*

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2 Consider a 2-component real vector field $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ and its Lagrangian density in 3+1 space-time dimensions,

$$\mathcal{L} = (\partial_\mu \boldsymbol{\phi})^T M (\partial^\mu \boldsymbol{\phi}) - \boldsymbol{\phi}^T M \boldsymbol{\phi} - \lambda \left(\boldsymbol{\phi}^T M \boldsymbol{\phi}\right)^2 ,$$

where M is a real symmetric matrix and $\lambda > 0$.

(a) [**bookwork**] Derive the Euler-Lagrange equations of motion for the fields ϕ_1 and ϕ_2. [5]

Using the general Euler-Lagrange formula for a multi-component field [2]

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} ,$$

and taking advantage of the symmetric form of M, $M_{ij} = M_{ji}$, we obtain $\sum_j M_{ij} \left[\partial_\mu \partial^\mu \phi_j + \phi_j + 2\lambda(\boldsymbol{\phi}^T M \boldsymbol{\phi})\phi_j\right] = 0$, for $i = 1, 2$. [3]

(b) [**bookwork**] State Noether's theorem and write a general expression for the conserved current in the case of a multi-component field. [3]

For every *continuous* symmetry of the Lagrangian density (namely, a continuous transformation that leave $\mathcal{L}$ unchanged), there is a conserved current [2]

$$J^\mu = \sum_{i=1}^{4} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i, \qquad \partial_\mu J^\mu = 0 ,$$

where $\delta\phi_i$ is the infinitesimal change in the field ϕ_j under the symmetry transformation. [1]

(c) [**unseen calculation**] Consider the transformation $\boldsymbol{\phi} \to D\boldsymbol{\phi}$ where $D = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}$ with $\alpha > 0$. Find the conditions that one must impose on the elements of the matrix M so that this transformation is a symmetry of the system. [5]

From the form of the Lagrangian density above, we see that it is left invariant under the transformation $\boldsymbol{\phi} \to D\boldsymbol{\phi}$ if and only if $D^T M D = M$. Namely,

$$D^T M D = \begin{pmatrix} \alpha^2 a & b \\ b & c/\alpha^2 \end{pmatrix} = M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} ,$$

which can be satisfied for $\alpha > 0$ only if $a = c = 0$ and $M = b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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(d) [**part bookwork, part unseen calculation**] Show that, under such conditions, the symmetry of the system with respect to the transformation $\boldsymbol{\phi} \to D\boldsymbol{\phi}$ leads to a conserved current of the form [5]

$$J^\mu \propto \phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1 \,.$$

Under such conditions, the Lagrangian density takes the form [1]

$$\mathcal{L} = 2b\,(\partial_\mu\phi_1)\,(\partial^\mu\phi_2) - 2b\,\phi_1\phi_2 - 4b^2\lambda\,(\phi_1\phi_2)^2 \,.$$

In order to apply Noether's theorem, we firstly ought to expand the continuous transformation to leading order about the identity, $\alpha = 1+\varepsilon$, and obtain the small variations in the field components: [2]

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1/(1+\varepsilon) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \simeq \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \varepsilon \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix} ,$$

for ε small. One can then substitute into the expression for the current at point (c):

$$J^\mu \;=\; \sum_{i=1}^{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \propto \phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1$$

where we have dispensed with the trivial overall constant factor of $2b\varepsilon$. [2]

(e) [**part bookwork; part unseen**] Using the Fourier representation for each component of the field,

$$\phi_i = \int \mathrm{d}^3k\; N(\boldsymbol{k})\left[a_i(\boldsymbol{k})e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}} + a_i^*(\boldsymbol{k})e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}\right] , \qquad\qquad i = 1,2\,,$$

where $N(\boldsymbol{k}) = [(2\pi)^3 2\omega]^{-1}$, find the dispersion relation between ω and $\boldsymbol{k}$ when $\lambda = 0$ by substituting into the equations of motion. Then express the conserved charge associated with the current $J^\mu = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1$ in terms of the relevant Fourier modes $a_i(\boldsymbol{k})$. [7]

Note that the time derivative operator ∂_0 acting on the Fourier representation of ϕ_i leads to a factor of $i\omega$ in front of the a_i term, and to a factor of $-i\omega$ in front of the a_i^* term. Similarly, the space derivative operator ∂_j leads to the factors $-ik_j$ and ik_j, respectively.

Therefore, the Euler-Lagrange equations obtained in (b) after setting $\lambda = 0$, $\sum_j M_{ij}\left[\partial_\mu\partial^\mu\phi_j + \phi_j\right] = 0$, can be written as

$$\int \mathrm{d}^3k\; N(\boldsymbol{k})\left(-\omega^2 + k^2 + 1\right)\sum_j M_{ij}\left[a_j(\boldsymbol{k})e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}} + a_j^*(\boldsymbol{k})e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}\right] = 0\,,$$

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which in turn give the relationship $\omega^2 = k^2 + 1$ for all $i = 1, 2$. [3]

The conserved charge associated with Noether's current J^μ is $Q = \int J^0 \mathrm{d}r^3 = \int [\phi_1 \partial_t \phi_2 - \phi_2 \partial_t \phi_1] \mathrm{d}^3 r$. [1]

Let us first consider

$$\phi_1 \partial_t \phi_2 = \int \mathrm{d}^3k \mathrm{d}^3 \, N(\boldsymbol{k}) N(\boldsymbol{k}') \; \left[a_1(\boldsymbol{k}) e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}} + a_1^*(\boldsymbol{k}) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \left[i\omega a_2(\boldsymbol{k}') e^{i\omega' t - i\boldsymbol{k}'\cdot\boldsymbol{x}} - i\omega a_2^*(\boldsymbol{k}') e^{-i\omega' t + i\boldsymbol{k}'\cdot\boldsymbol{x}}\right] .$$

Integrating this term in $\mathrm{d}^3 r$, and recalling that $N(\boldsymbol{k}) = N(-\boldsymbol{k})$ and $\omega(\boldsymbol{k}) = \omega(-\boldsymbol{k})$, one obtains

$$\int \phi_1 \partial_t \phi_2 \, \mathrm{d}^3 r = \int \mathrm{d}^3 k \; N^2(\boldsymbol{k}) i\omega(\boldsymbol{k}) \left[a_1(\boldsymbol{k}) a_2(-\boldsymbol{k}) e^{2i\omega t} - a_1^*(\boldsymbol{k}) a_2^*(-\boldsymbol{k}) e^{-2i\omega t} - a_1(\boldsymbol{k}) a_2^*(\boldsymbol{k}) + a_1^*(\boldsymbol{k}) a_2(\boldsymbol{k})\right] .$$

The other term, $\int \phi_2 \partial_t \phi_1 \, \mathrm{d}^3 r$ differs only by swapping the indices $1 \leftrightarrow 2$ and by an overall minus sign in the expression for Q. Therefore, when summed together, one obtains

$$\begin{aligned} Q &= \int \mathrm{d}^3 k \; N^2(\boldsymbol{k}) i\omega(\boldsymbol{k}) \times \\ &\quad \Big\{ \left[a_1(\boldsymbol{k}) a_2(-\boldsymbol{k}) - a_2(\boldsymbol{k}) a_1(-\boldsymbol{k})\right] e^{2i\omega t} - \left[a_1^*(\boldsymbol{k}) a_2^*(-\boldsymbol{k}) - a_2^*(\boldsymbol{k}) a_1^*(-\boldsymbol{k})\right] e^{-2i\omega t} \\ &\quad - a_1(\boldsymbol{k}) a_2^*(\boldsymbol{k}) + a_1^*(\boldsymbol{k}) a_2(\boldsymbol{k}) + a_2(\boldsymbol{k}) a_1^*(\boldsymbol{k}) - a_2^*(\boldsymbol{k}) a_1(\boldsymbol{k}) \Big\} \\ &= \int \mathrm{d}^3 k \; N^2(\boldsymbol{k}) 2i\omega(\boldsymbol{k}) \left\{ a_1^*(\boldsymbol{k}) a_2(\boldsymbol{k}) - a_1(\boldsymbol{k}) a_2^*(\boldsymbol{k}) \right\} , \end{aligned}$$

where we used the fact that $\boldsymbol{k}$ is an integration variable and, upon changing $\boldsymbol{k} \to -\boldsymbol{k}$, the terms in square brackets can be shown to cancel. (If this point is missed in the solutions, only one mark will be deducted.) [3]

Bonus mark if the solution remarks that the quantity in curly brackets is purely imaginary and therefore the conserved charge as defined in the question is a real scalar quantity.

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3 Consider a real scalar field $\phi(t,x)$ in $1+1$ space-time dimensions, with action $S = \int \mathrm{d}t\,\mathrm{d}x\,\mathcal{L}$ and Lagrangian density

$$\mathcal{L} = \dot{\phi}^2 - \gamma\,\phi'^2 + \alpha\,\phi^2 - \frac{\beta}{2}\,\phi^4\,,$$

where $\dot{\phi} = \frac{\partial\phi}{\partial t}$ and $\phi' = \frac{\partial\phi}{\partial x}$, and α, β, γ are real and positive constants.

(a) [**part bookwork, part unseen calculation**] Find the units of α, β and γ, and use them to obtain a characteristic length scale and an energy scale for the system. [7]

Since we consider the field in $1+1$ space dimensions, the units of the Lagrangian density are energy $[E]$ divided by length $[L]$ (or in the natural units $\hbar = 1 = c$ we have $[\mathcal{L}] = [M]^2$). [1]

Since $[\dot{\phi}] = [\phi][T]^{-1}$ and $[\mathcal{L}] = [\dot{\phi}]^2$, we obtain the units of the field $[\phi] = [E]^{1/2}[L]^{-1/2}[T]$ (or the field is dimensionless $[\phi] = [M]^0$). [1]

It then follows that

$$[\alpha] = [T]^{-2}\,, \quad [\beta] = [E]^{-1}[L][T]^{-4}\,, \quad [\gamma] = [L]^2[T]^{-2}$$

(or $[\alpha] = [M]^2 = [\beta]$ and $[\gamma] = [M]^0$). [3]

Therefore, we can obtain a characteristic length scale, by considering a unit of length in terms of the parameters of the system, [1]

$$\lambda = \left(\frac{\gamma}{\alpha}\right)^{1/2}$$

(up to a multiplicative constant), and, similarly, by considering a unit of energy, a characteristic energy scale [1]

$$\epsilon = \beta^{-1}\left(\frac{\gamma}{\alpha}\right)^{1/2}\alpha^2 = \frac{\gamma^{1/2}\alpha^{3/2}}{\beta}$$

(up to a multiplicative constant). *[λ and ϵ will describe the width and the energy of the interface discussed in (c) and (d).]*

(b) [**bookwork**] Derive the components of the stress-energy tensor, and discuss its conservation. Use it to define the total energy E of the system. [8]

The stress-energy tensor is defined as [1]

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\,.$$

(or equivalently $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$).

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Considering the relativistic metric $g^{\mu\nu} = (-1)^\mu \delta_{\mu\nu}$, we have $\partial_0 \phi = \dot{\phi} = \partial^0 \phi$ and $\partial_1 \phi = \phi' = -\partial^1 \phi$, and the components of the stress-energy tensor are: $T^{00} = \dot{\phi}^2 + \gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4$, $T^{01} = -2\dot{\phi}\phi'$, $T^{10} = -2\gamma\,\phi'\dot{\phi}$, and $T^{11} = \dot{\phi}^2 + \gamma\,\phi'^2 + \alpha\,\phi^2 - \frac{\beta}{2}\,\phi^4$. *[Notice that the tensor is not symmetric unless* $\gamma = 1$*, while* $T^{00} = \mathcal{H} \neq T^{11}$ *unless* $\alpha = 0 = \beta$*.]* [4]

Alternatively, for the metric $g^{\mu\nu} = \delta_{\mu\nu}$, we have $\partial_0 \phi = \dot{\phi} = \partial^0 \phi$ and $\partial_1 \phi = \phi' = \partial^1 \phi$ and the components are: $T^{00} = \dot{\phi}^2 + \gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4$, $T^{01} = 2\dot{\phi}\phi'$, $T^{10} = -2\gamma\,\phi'\dot{\phi}$, and $T^{11} = -\dot{\phi}^2 - \gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4$.

The stress-energy tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$, since the Lagrangian density is independent from the time coordinate ($\partial_\mu T^{\mu 0} = 0$) and the space coordinate ($\partial_\mu T^{\mu 1} = 0$), that is the action is invariant under time-space translations. Alternatively, the conservation for the system can be checked by a direct calculation. [2]

The total energy *density* of the system is the T^{00} component of the stress-energy tensor. (Alternatively, it can be derived as in the lecture notes, $\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L}$.) The total energy of the system is the integral [1]

$$E = \int \mathrm{d}x\,\mathcal{H} = \int \mathrm{d}x\,\left(\dot{\phi}^2 + \gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4\right).$$

[Notice that the total energy corresponds to a conserved charge, when the energy flux T^{10} *is* 0 *at* $x \to \pm\infty$*, e.g. for an asymptotically uniform field, as considered in (c).]*

(c) [**unseen calculation**] Consider a field that interfaces between the two constant values:

$$\lim_{x\to-\infty} \phi(x,t) = -\sqrt{\frac{\alpha}{\beta}} \qquad \text{and} \qquad \lim_{x\to+\infty} \phi(x,t) = \sqrt{\frac{\alpha}{\beta}}.$$

Using an appropriate variational calculation ($\phi \to \phi + \delta\phi$), or otherwise, show that the field $\phi(x,t)$ which minimises the total energy E for the above boundary conditions takes the form

$$\phi(x) = \sqrt{\frac{\alpha}{\beta}}\tanh\left[\sqrt{\frac{\alpha}{2\gamma}}(x - x_0)\right],$$

where x_0 is an arbitrary constant. (It is sufficient to show that $\delta E = 0$ and you do not need to demonstrate that it is an actual minimum.) [10]

We need to find the field $\phi(x,t)$ that minimises the total energy E of the system. Since the boundary condition is time-independent, the energy is minimised when $\dot{\phi} = 0$, namely for a field $\phi = \phi(x)$ not dependent on time. [1]

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We are therefore left with the minimisation of

$$E = \int \mathrm{d}x \, \left(\gamma \, \phi'^2 - \alpha \, \phi^2 + \frac{\beta}{2} \, \phi^4 \right) \, ,$$

with boundary conditions

$$\lim_{x \to \pm\infty} \phi(x) = \pm \sqrt{\frac{\alpha}{\beta}} \, .$$

The task at hand is equivalent to the minimisation of the action in Hamilton's principle in the lecture notes, barring the fact the integration variable is x instead of t. We can therefore proceed in the same way by considering the variation in E induced by a variation in $\phi \to \phi + \delta\phi$ and $\phi' \to \phi' + (\delta\phi)'$. [1]

Computing the resulting variation $\delta E = E(\phi + \delta\phi, \phi' + (\delta\phi)') - E(\phi, \phi')$ to leading order and cancelling the trivial zeroth order terms, we obtain [2]

$$\delta E = 2 \int \mathrm{d}x \, \left[\gamma \, \phi' (\delta\phi)' - \alpha \, \phi \delta\phi + \beta \, \phi^3 \delta\phi \right] \, .$$

The last term can be rewritten as $\phi'(\delta\phi)' = (\phi'\delta\phi)' - \phi''\delta\phi$; out of the two terms on the right hand side, the former contributes a boundary term to the integral:

$$(\phi'\delta\phi)|_{x=-\infty}^{x=+\infty} = 0 \, ,$$

which is 0 due to the variation disappearing at the boundary, $\lim_{x\to\pm\infty} \delta\phi = 0$ (given the choice of boundary conditions). We are therefore left only with the latter term: [2]

$$\delta E = 2 \int \mathrm{d}x \, \left(-\gamma \, \phi'' - \alpha \, \phi + \beta \, \phi^3 \right) \delta\phi \, .$$

We are looking to impose $\delta E = 0$ for a completely arbitrary choice of the variation function $\delta\phi$, and this requires

$$-\gamma \, \phi'' - \alpha \, \phi + \beta \, \phi^3 = 0$$

identically throughout space. [1]

Alternatively, by extending the analogy to the action principle, one can see that the energy integral is extremised by the corresponding 'Euler-Lagrange' equations with respect to 'time' x

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial E}{\partial \phi'} = \frac{\partial E}{\partial \phi} \, ,$$

which again gives

$$\gamma \, \phi'' = -\alpha \, \phi + \beta \, \phi^3 .$$

Method 1: We can now verify that

$$\phi(x) = \sqrt{\frac{\alpha}{\beta}} \tanh \left[\sqrt{\frac{\alpha}{2\gamma}} (x - x_0) \right] \, ,$$

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fulfils the equation, as [2]

$$\gamma\,\phi''(x) = \gamma\sqrt{\frac{\alpha^2}{2\beta\gamma}}\,\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cosh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]} = -\alpha\,\frac{\sqrt{\frac{\alpha}{\beta}}\tanh\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]}{\cosh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]}\,,$$

while

$$-\alpha\,\phi(x)+\beta\,\phi(x)^3 = -\alpha\,\phi(x)\left\{1-\tanh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]\right\} = -\alpha\,\frac{\phi(x)}{\cosh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]}.$$

Finally, since $\lim_{x\to\pm\infty}\tanh[\sqrt{\alpha/(2\gamma)}(x-x_0)] = \pm 1$, the boundary conditions are satisfied. [1]

[We see that the width of the interface is proportional to the characteristic length scale λ. The solution is not unique due to the arbitrary choice of x_0, which is not fixed by the boundary conditions nor by other requirements. This corresponds to the 'centre' of the interface, namely the position x where $\phi(x) = 0$. The arbitrary parameter is a consequence of the fact that the asymptotic phases have exactly the same energy density, and the ratio of the space occupied by one or the other is not fixed by the requirement of minimizing the energy.]

Method 2: Here we provide full derivation of the solution for completeness.

In order to solve the differential equation on the field, it is convenient to multiply all terms by ϕ' and rearrange:

$$-\gamma\,\phi''\phi' - \alpha\,\phi\phi' + \beta\,\phi^3\phi' = 0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(-\gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4\right).$$

The term in brackets is thence a constant in space, and we can find it using the boundary conditions $\lim_{x\to\pm\infty}\phi(x) = \pm\sqrt{\frac{\alpha}{\beta}}$ and $\lim_{x\to\pm\infty}\phi(x)' = 0$: [1]

$$\gamma\,\phi'^2 = -\alpha\,\phi^2 + \frac{1}{2}\beta\,\phi^4 + \frac{\alpha^2}{2\beta} = \frac{\beta}{2}\left(\phi^2 - \frac{\alpha}{\beta}\right)^2.$$

Alternatively, again using the analogy of E to an action, the 'energy' of such fictitious system,

$$\gamma\,\phi'^2 + \alpha\,\phi^2 - \frac{\beta}{2}\,\phi^4\,,$$

is conserved and therefore independent of x.

From the boundary conditions at $x\to\pm\infty$, we have that $\phi'\to 0$ and the 'energy' is $\alpha^2/(2\beta)$, not dependent on x. This gives the following relation obeyed by the solution of the equations of motion above: [1]

$$\gamma\,\phi'^2 = -\alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4 + \frac{\alpha^2}{2\beta} = \frac{\beta}{2}\left(\phi^2 - \frac{\alpha}{\beta}\right)^2.$$

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From the boundary conditions, the sign of the gradient is positive and we arrive at the equation

$$\phi' = -\sqrt{\frac{\beta}{2\gamma}}\left(\phi^2 - \frac{\alpha}{\beta}\right),$$

which by introducing $\tilde{\phi} = \phi/\sqrt{\alpha/\beta}$ simplifies to

$$\tilde{\phi}' = \sqrt{\frac{\alpha}{2\gamma}}\left(1 - \tilde{\phi}^2\right) = \frac{1}{\sqrt{2}\lambda}\left(1 - \tilde{\phi}^2\right),$$

where λ is the characteristic length scale derived in (a). Using integration by separation of variables we get

$$\int_0^{\tilde{\phi}(x)} \frac{\mathrm{d}\tilde{\phi}}{1 - \tilde{\phi}^2} = \frac{1}{\sqrt{2}\lambda}\int_{x_0}^{x} \mathrm{d}x,$$

where we assumed $\tilde{\phi}(x_0) = 0$. We can directly calculate the left integral by observing that

$$\int_0^{\tilde{\phi}(x)} \frac{\mathrm{d}\tilde{\phi}}{1 - \tilde{\phi}^2} = \frac{1}{2}\int_0^{\tilde{\phi}(x)} \mathrm{d}\tilde{\phi}\left(\frac{1}{1 - \tilde{\phi}} + \frac{1}{1 + \tilde{\phi}}\right)$$

($|\tilde{\phi}| < 1$) and we arrive at

$$\frac{1}{2}\ln\left[\frac{1 + \tilde{\phi}(x)}{1 - \tilde{\phi}(x)}\right] = \frac{1}{\sqrt{2}\lambda}(x - x_0).$$

Thus, [1]

$$\phi(x) = \sqrt{\frac{\alpha}{\beta}}\,\tilde{\phi}(x) = \sqrt{\frac{\alpha}{\beta}}\,\frac{-1 + \exp\left[\frac{\sqrt{2}}{\lambda}(x - x_0)\right]}{1 + \exp\left[\frac{\sqrt{2}}{\lambda}(x - x_0)\right]} = \sqrt{\frac{\alpha}{\beta}}\tanh\left(\frac{x - x_0}{\sqrt{2}\lambda}\right).$$

(d) [**unseen calculation**] Define the energy E_I of the interface as the difference between the energy of the field discussed in point (c), namely with the interface present, and the energy of a uniform field, $\phi = \sqrt{\alpha/\beta}$. Either by direct computation or by an appropriate scaling analysis, determine how E_I depends on the parameters α, β and γ.

[Up to 5 bonus marks]

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2 bonus marks for noting that the energy density of the uniform phase is $-\alpha^2/(2\beta)$.

3 bonus marks for using this information and noting that

$$\phi'(x) = \frac{\alpha}{\sqrt{2\beta\gamma}\,\cosh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]}\,, \qquad \gamma[\phi'(x)]^2 = \frac{\alpha^2}{2\beta\,\cosh^4\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]}\,,$$

while

$$\begin{aligned}
-\alpha\,\phi(x)^2 + \frac{\beta}{2}\,\phi(x)^4 + \frac{\alpha^2}{2\beta} &= \frac{\alpha^2}{2\beta}\left\{-2\tanh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right] + \tanh^4\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right] + 1\right\} \\
&= \frac{\alpha^2}{2\beta}\left\{1 - \tanh^2\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]\right\}^2 \\
&= \frac{\alpha^2}{2\beta\,\cosh^4\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]},
\end{aligned}$$

to obtain

$$\begin{aligned}
E_I &= \int_{-\infty}^{+\infty} \mathrm{d}x\,\left(\gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4 + \frac{\alpha^2}{2\beta}\right) = 2\gamma\int_{-\infty}^{+\infty}\mathrm{d}x\ \phi'^2 \\
&= 2\gamma\int_{-\sqrt{\frac{\alpha}{\beta}}}^{\sqrt{\frac{\alpha}{\beta}}}\mathrm{d}\phi\ \phi' = \sqrt{2\beta\gamma}\int_{-\sqrt{\frac{\alpha}{\beta}}}^{\sqrt{\frac{\alpha}{\beta}}}\mathrm{d}\phi\ \left(\frac{\alpha}{\beta} - \phi^2\right) = \sqrt{2\beta\gamma}\left(\frac{\alpha}{\beta}\phi - \frac{1}{3}\phi^3\right)\Big|_{-\sqrt{\frac{\alpha}{\beta}}}^{\sqrt{\frac{\alpha}{\beta}}} \\
&= \frac{4\sqrt{2}}{3}\beta^{-1}\gamma^{1/2}\alpha^{3/2} = \frac{4\sqrt{2}}{3}\epsilon\,.
\end{aligned}$$

The energy of the interface is finite and proportional to the characteristic energy scale ϵ. *As $\alpha \to 0^+$ the interface energy E_I tends to zero, as expected when the two uniform phases $\pm\sqrt{\alpha/\beta}$ merge into a single phase $\phi = 0$.*

Alternatively,

$$\begin{aligned}
E_I &= \int_{-\infty}^{+\infty}\mathrm{d}x\,\left(\gamma\,\phi'^2 - \alpha\,\phi^2 + \frac{\beta}{2}\,\phi^4 + \frac{\alpha^2}{2\beta}\right) = \frac{\alpha^2}{\beta}\int_{-\infty}^{+\infty}\mathrm{d}x\ \frac{1}{\cosh^4\left[\sqrt{\frac{\alpha}{2\gamma}}(x-x_0)\right]} \\
&= \frac{\sqrt{\alpha^3\gamma}}{\beta}\int_{-\infty}^{+\infty}\mathrm{d}y\ \frac{\sqrt{2}}{\cosh^4(y)} = \epsilon\,A\,,
\end{aligned}$$

where we introduced $y = \sqrt{\frac{\alpha}{2\gamma}}(x-x_0)$, and A is dimensionless, positive and finite as $1/\cosh^4(y) \sim 16\,e^{\mp 4y}$ for $y \to \pm\infty$.

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4 Consider the following Lagrangian density for a complex relativistic scalar field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) + \frac{m^2}{2}\left(\phi^*\phi\right)^2 - \frac{\lambda}{3}\left(\phi^*\phi\right)^3 ,$$

where m and λ are real positive constants.

(a) **[part book work, part new]** Derive the minimal energy state(s) of the field ϕ and obtain the Lagrangian density for small fluctuations χ about (one of) the minimum energy state(s), ϕ_0, up to quadratic order in χ. Discuss briefly the behaviour of the real and imaginary components of χ. [8]

The minimal energy states correspond to a field ϕ_0 constant throughout spacetime. In order to find its value, let us set $x = \phi_0^*\phi_0 \geq 0$ and minimise the remaining terms in the energy,

$$V(x) = -\frac{m^2}{2}x^2 + \frac{\lambda}{3}x^3 \qquad \frac{dV}{dx} = -m^2 x + \lambda x^2 = 0 \qquad \to \qquad x = 0, \frac{m^2}{\lambda} .$$

From the form of $V(x)$ it is straightforward to observe that $x = m^2/\lambda$ corresponds to the minimum of the energy. This gives a continuously degenerate set of minima: $\phi_0 = \sqrt{m^2/\lambda}\, e^{i\theta}$, $\theta \in [0, 2\pi)$. [3]

Bonus mark for observing that, at low energies, the system breaks spontaneously the rotational (phase) symmetry and chooses one of these minima. Fluctuations about any such minimum are highly anisotropic.

Without loss of generality (by rotational symmetry) we can choose to study the small fluctuations about any given such minima, say $\phi_0 = \sqrt{m^2/\lambda}$ for convenience, and consider $\phi = \phi_0 + \chi$ up to quadratic order in χ.

Firstly, let us write $(\phi_0 + \chi)^*(\phi_0 + \chi) = \phi_0^2[1 + (\chi^* + \chi)/\phi_0 + \chi^*\chi/\phi_0^2]$, which is of the form $1 + \epsilon$ with ϵ small (linear in χ). And we use the expansions $(1 + \epsilon)^2 = 1 + 2\epsilon + \epsilon^2$ and $(1 + \epsilon)^3 \simeq 1 + 3\epsilon + 3\epsilon^2$ (where further terms were neglected as they are of higher than second order in χ). Retaining only up to second order terms, we obtain

$$\begin{aligned}
\left[(\phi_0 + \chi)^*(\phi_0 + \chi)\right]^2 &\simeq \phi_0^4 + 2\phi_0^3(\chi^* + \chi) + 2\phi_0^2\chi^*\chi + \phi_0^2(\chi^* + \chi)^2 \\
\left[(\phi_0 + \chi)^*(\phi_0 + \chi)\right]^3 &\simeq \phi_0^6 + 3\phi_0^5(\chi^* + \chi) + 3\phi_0^4\chi^*\chi + 3\phi_0^4(\chi^* + \chi)^2 .
\end{aligned}$$

We then combine the two terms in $V(\phi_0 + \chi)$, remembering that the first term is multiplied by $m^2/2$ and the second by $-\lambda/3 = -m^2/(3\phi_0^2)$. Several terms cancel and we arrive at the result

$$V(\phi_0 + \chi) \simeq \frac{m^2\phi_0^4}{6} - \frac{m^2\phi_0^2}{2}(\chi^* + \chi)^2 + \mathcal{O}(\chi^3) .$$

(Note that $m^2\phi_0^2 = m^4/\lambda$.)

Finally, the desired Langrangian density for small fluctuations reads

$$\mathcal{L} = (\partial_\mu \chi^*)(\partial^\mu \chi) - \frac{m^4}{2\lambda}\left(\chi^* + \chi\right)^2 ,$$

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up to irrelevant constants and terms of order χ^3 and higher. [4]

Only the real part of the fluctuating field (radial component) has a *positive* mass term; the imaginary part (tangential component) is massless and is conventionally referred to as the Goldstone mode produced by the breaking of a continuous symmetry. [1]

(b) [**book work**] Consider coupling the complex scalar field ϕ in this question to an electromagnetic field via the covariant derivative. Discuss briefly what happens to the fluctuating complex field χ as a result. [3]

When the complex field is coupled to an electromagnetic field through the covariant derivative, the massless Goldstone mode gets absorbed by the electromagnetic field, which as a result acquires a mass. [1]

The massive (real component) mode of the fluctuating complex field decouples (to second order) from the electromagnetic field, and it is usually referred to as the Higgs mode (boson). [2]

(c) [**part book work, part new**] Write the Lagrangian density of such a coupled electromagnetic field when the complex scalar field is exactly at (one of) its minimum energy state(s). If needed, you may ignore irrelevant constant terms. Find the corresponding Euler-Lagrange equations of motion for the 4-vector potential, and show that (by an appropriate choice of gauge or otherwise) they can be written as [6]

$$\left(\partial_t^2 - \partial_x^2 + \frac{2e^2m^2}{\lambda}\right) A^\nu = 0\,,$$

in 1+1 space-time dimensions and natural units. You may use, without deriving it, the result:

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(F_{\alpha\beta}F^{\alpha\beta}\right) = 4F^{\mu\nu} \qquad \text{where} \qquad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha\,.$$

Proceeding as instructed, we introduce the covariant derivative, $D_\mu = \partial_\mu + ieA_\mu$, and consider

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) + \frac{m^2}{2}(\phi^*\phi)^2 - \frac{\lambda}{3}(\phi^*\phi)^3\,.$$

Substituting $\phi = \phi_0 = \sqrt{m^2/\lambda}$, we arrive at the electromagnetic Lagrangian density

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (ieA_\mu\phi_0)^*(ieA^\mu\phi_0) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\phi_0^*\phi_0 A_\mu A^\mu \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2m^2}{\lambda}A_\mu A^\mu\,,\end{aligned}$$

(TURN OVER

where the last equality holds up to irrelevant constant terms. [2]

The Euler-Lagrange equations of motion for this electromagnetic Lagrangian are

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\,,$$

and after a few lines of algebra, one obtains [2]

$$\frac{2e^2m^2}{\lambda} A^\nu = -\partial_\mu F^{\mu\nu} = -\partial_\mu \partial^\mu A^\nu + \partial_\mu \partial^\nu A^\mu\,.$$

At this point one can choose to work in the Lorentz gauge, $\partial^\mu A^\mu = 0$, and drop the last term in the equation above. Alternatively, applying ∂_ν to the left and right hand side of the first equality in the equation above, one obtains $(2e^2m^2/\lambda)\,\partial_\nu A^\nu = -\partial_\nu \partial_\mu F^{\mu\nu} = 0$, by contraction of a symmetric with an antisymmetric tensor; this in fact requires that $\partial_\nu A^\nu = 0$.

Finally, in 1+1 space-time dimensions, we arrive at the equations of motion for the 4-vector potential: [2]

$$\begin{aligned} \frac{2e^2m^2}{\lambda} A^\nu &= -\partial_\mu \partial^\mu A^\nu \\ \left(\partial_t^2 - \partial_x^2 + \frac{2e^2m^2}{\lambda}\right) A^\nu &= 0\,. \end{aligned}$$

(d) **[new]** Using the Fourier transform conventions:

$$\tilde{A}^\nu(k,t) = \int \hat{A}^\nu(k,\omega)\, e^{-i\omega t}\, \frac{\mathrm{d}\omega}{2\pi} \qquad\qquad A^\nu(x,t) = \iint \hat{A}^\nu(k,\omega)\, e^{-ikx-i\omega t}\, \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2}\,,$$

and defining the constant $M^2 = 2e^2m^2/\lambda$ for convenience, derive the Green's function $\tilde{A}^\nu(k,t)$ for the equations of motion in (c). (This may require shifting poles or deforming the integration contour, according to the physical expectation in a relativistic system.) [8]

Using these Fourier conventions and the equation of motion obtained previously, we find that [1]

$$\Big(-\omega^2 + k^2 + M^2\Big)\, \hat{A}^\nu(k,\omega) = 1\,,$$

and from it [1]

$$\tilde{A}^\nu(k,t) = \int \frac{e^{-i\omega t}}{-\omega^2 + k^2 + M^2}\, \frac{\mathrm{d}\omega}{2\pi}\,.$$

We recognise this as an integral that can be computed using Cauchy's theorem. The integrand has two poles along the real axis which we deal with as appropriate for a relativistic theory by introducing an imaginary shift ($\varepsilon > 0$):

$$-\omega^2 + k^2 + M^2 + i\varepsilon\,,$$

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leading to the complex roots: $\omega_{1,2} = \pm\sqrt{k^2+M^2} \pm i\varepsilon$ (where we expanded in small ε, which we will eventually take to zero, and we absorbed a factor of $[2\sqrt{k^2+M^2}]^{-1}$ into the definition of ε). [2]

For $t > 0$ we close the contour in the lower half plane (clockwise); for $t < 0$ we close the contour in the upper half plane (counterclockwise), and we find that: [2]

$$\tilde{A}^\nu(k,t) = \int \frac{-e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} \frac{\mathrm{d}\omega}{2\pi} = \begin{cases} i\frac{e^{-i\omega_2 t}}{\omega_2-\omega_1} & (t>0) \\ -i\frac{e^{-i\omega_1 t}}{\omega_1-\omega_2} & (t<0) \end{cases} .$$

Substituting $\omega_{1,2}$ into $\tilde{A}^\nu(k,t)$ and sending $\varepsilon \to 0$ we arrive at the final result [2]

$$\tilde{A}^\nu(k,t) = \begin{cases} -i\frac{e^{i\sqrt{k^2+M^2}\,t}}{2\sqrt{k^2+M^2}} & (t>0) \\ -i\frac{e^{-i\sqrt{k^2+M^2}\,t}}{2\sqrt{k^2+M^2}} & (t<0) \end{cases} = \frac{-i\,e^{i\sqrt{k^2+M^2}\,|t|}}{2\sqrt{k^2+M^2}} .$$

Bonus mark if one notes that the final result is time-reversal symmetric, as expected for relativistic systems.

END OF PAPER