NATURAL SCIENCES TRIPOS Part II

Wednesday 16 January 2019 10:30am to 12:30pm

THEORETICAL PHYSICS I

Answer **all four** *questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.

1 The action for a system consisting of a polarizable medium moving relativistically in an electromagnetic field is given by

$$S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^{\mu\nu} F_{\mu\nu} - \frac{1}{2\kappa} u_\mu u_\nu M^{\lambda\mu} M^{\nu}{}_{\lambda} + \frac{1}{4\chi} (g_{\mu\rho} + u_\mu u_\rho)(g_{\nu\sigma} + u_\nu u_\sigma) M^{\mu\nu} M^{\sigma\rho} \right],$$

where $F^{\mu\nu}$ is the usual electromagnetic field strength tensor, $g^{\mu\nu}$ is the usual Minkowski metric, $M^{\mu\nu}$ is an antisymmetric tensor describing the polarization of the medium, u^μ is the 4-velocity of the medium, and κ and χ are physical constants.

(a) Show that the equation of motion for the electromagnetic gauge potential, A_μ, is given by [5]

$$\partial_\nu F^{\mu\nu} = \partial_\nu M^{\mu\nu}.$$

(b) Using the method of Lagrange multipliers to account for the antisymmetric nature of $M^{\mu\nu}$, show that the equation of motion for the antisymmetric polarization tensor, $M^{\mu\nu}$ is given by [10]

$$0 = F_{\mu\nu} - \frac{1}{\kappa}(u_\nu u_\rho M^{\rho}{}_{\mu} - u_\mu u_\rho M^{\rho}{}_{\nu}) - \frac{1}{\chi}\left[M^{\rho\sigma}(g_{\rho\mu} + u_\rho u_\mu)(g_{\sigma\nu} + u_\sigma u_\nu) \right].$$

(c) Defining $P_i \equiv M^{i0}$ and $M_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$, respectively, express the interaction term $\frac{1}{2} M^{\mu\nu} F_{\mu\nu}$ in terms of the usual electric and magnetic field vectors and thus give an interpretation of P_i and M_i. [4]

(d) Show that, in the rest frame, P_i and M_i are proportional to the electric and magnetic fields, respectively, and find the constants of proportion. [4]

(e) What properties of the medium do the constants κ and χ describe? [2]

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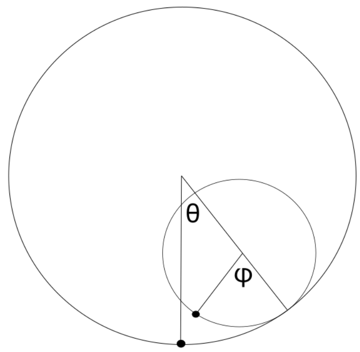
2 Describe what is meant by a *holonomic constraint* in Lagrangian classical mechanics and give an example of a system with a holonomic constraint. [4]

A uniform cylinder of mass m and radius r rolls without slipping inside a larger cylinder of radius R. The angles θ and ϕ are defined as in the Figure, where the black dots coincide when the rolling cylinder is at its lowest point.



By resolving the motion into the motion of the centre of mass and motion about the centre of mass, show that the kinetic energy may be written as [5]

$$\frac{m}{2}(R-r)^2\dot{\theta}^2 + \frac{m}{4}r^2(\dot{\phi}-\dot{\theta})^2\,.$$

Show that the no slip condition yields a holonomic constraint. [3]

By writing down the Lagrangian, show that the system can exhibit small oscillations and find the corresponding oscillation frequency. [6]

Discuss whether or not there is a frictional force present and whether or not the energy of the system is conserved. [3]

Discuss whether or not angular momentum is conserved. [2]

Discuss whether or not the system is holonomic for all possible motions. [2]

3 A Bose-Einstein condensate can described by the Lagrangian density

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2M}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi + \mu|\psi|^2 - \frac{1}{2}g|\psi|^4$$

for the complex field $\psi(\boldsymbol{r},t)$, with $g,\mu>0$.

(a) Show that this theory has symmetry under global changes of the phase of $\psi(\boldsymbol{r},t)$, but does not have symmetry under local phase changes. [3]

(b) Show that the Hamiltonian density is

$$\mathcal{H} = \frac{\hbar^2}{2M}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi - \mu|\psi|^2 + \frac{1}{2}g|\psi|^4\,.$$

Hence show that the groundstate, ψ_0, breaks the global symmetry. [5]

(c) By writing $\psi(\boldsymbol{r},t) = \psi_0 + \chi(\boldsymbol{r},t)$, with ψ_0 real, show that, keeping all terms up to second order in χ, the Lagrangian density may be written [7]

$$\mathcal{L} \simeq \frac{\mathrm{i}\hbar}{2}\left[\chi^*\frac{\partial\chi}{\partial t} - \chi\frac{\partial\chi^*}{\partial t}\right] - \frac{\hbar^2}{2M}\boldsymbol{\nabla}\chi^*\cdot\boldsymbol{\nabla}\chi - \frac{1}{2}\mu(\chi+\chi^*)^2\,.$$

(d) Derive the Euler-Lagrange equations for χ and χ^*. By writing $\chi = \chi_1 + \mathrm{i}\chi_2$, or otherwise, show that there exist excitations with angular frequency ω and wavevector $\boldsymbol{k}$ connected via [8]

$$\omega = \sqrt{\frac{|\boldsymbol{k}|^2}{2M}\left(\frac{\hbar^2|\boldsymbol{k}|^2}{2M} + 2\mu\right)}\,.$$

(e) Comment on the form of this mode dispersion in connection with Goldstone's theorem. [2]

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4 The Landau free energy describing a certain magnetic material is

$$f(m) = a(T - T_c)m^2 + \frac{1}{2}b\,m^4 + \frac{1}{3}c\,m^6$$

where the order parameter m is real, and $c > 0$.

(a) Explain the physical meaning of the Landau free energy and the order parameter. What symmetries does the above theory exhibit? [5]

(b) For $b > 0$, show that there is a continuous phase transition to an ordered phase for $T < T_c$. Determine the value of the critical exponent β, defined by $m \propto (T_c - T)^\beta$ for T close to T_c in the ordered phase. [8]

(c) Consider now $b < 0$. By sketching the free energy for $T = T_c$, or otherwise, show that the phase transition is now discontinuous. [4]

(d) Determine the critical temperature T_c^* at which this discontinuous transition occurs. [6]

[*Hint: it can be helpful to work in terms of* $x = m^2$.]

(e) Hence, or otherwise, sketch the phase diagram as a function of $T - T_c$ and of b, with both of these parameters ranging from negative to positive values. Be clear to distinguish continuous and discontinuous transitions. [2]

END OF PAPER