NATURAL SCIENCES TRIPOS Part II

Wednesday 18 January 2017 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **all** *questions to the best of your abilities. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A long elastic rod with radius a, bulk modulus K and density ρ is aligned along the z axis. The material originally at z is displaced to $z+\psi(z,t)$, leading to the Lagrangian density (per unit length)

$$\mathcal{L} = \frac{\pi a^2}{2}\rho\dot{\psi}^2 - \frac{\pi a^2}{2}K\psi'^2\,,$$

where $\dot{\psi} = \partial\psi/\partial t$ and $\psi' = \partial\psi/\partial z$.

(a) Find the canonical momentum conjugate to ψ, the Hamiltonian density (in terms of its proper variables) and the equation of motion of the system. Show that the equation of motion is satisfied by a wave-packet $\psi = f(z-ct)$, and find the corresponding value of c. [6]

(b) Show the action is invariant under (1) a uniform displacement of the rod, $\psi(z,t)\to\psi(z,t)+b$, and (2) a transformation of the waveform $\psi(z,t)\to\psi(bz,bt)$. Find the corresponding conserved quantities Q_1 and Q_2. [10]

(c) Show explicitly that Q_1 and Q_2 are conserved in a wavepacket $\psi = f(z-ct)$. [3]

(d) Discuss whether quantities corresponding to Q_1 and Q_2 are conserved in the following Lagrangian densities,

$$\begin{aligned}\mathcal{L}_1 &= \frac{\pi a^2}{2}\rho(z)\dot{\psi}^2 - \frac{\pi a^2}{2}K\psi'^2\\ \mathcal{L}_2 &= \frac{\pi a^2}{2}\rho\dot{\psi}^2 - \frac{\pi a^2}{2}K\psi'^2 - \alpha\psi'^4\\ \mathcal{L}_3 &= \frac{\pi a^2}{2}\rho\dot{\psi}^2 - \frac{\pi a^2}{2}K\psi'^2 - \beta\psi^2\psi'^2,\end{aligned}$$

and, if they are, whether they differ from Q_1 and Q_2 found in (b). [6]

2 A long solenoid with length l and radius a is centered on the z axis and generates a cylindrically symmetric magnetic field $\boldsymbol{B}(\rho, z)$, in cylindrical coordinates (ρ, θ, z). This field is uniform inside the solenoid, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, and negligible outside. A particle with mass m and charge q is fired towards the solenoid from a large distance away with large initial velocity $v\hat{\boldsymbol{z}}$ and initial radius $\rho_0 < a$.

(a) Show that the relativistic Lagrangian for the particle can be written as

$$L = -mc^2\sqrt{1 - \frac{\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\theta}^2}{c^2}} - V \quad \text{where} \quad V = \begin{cases} 0 & \text{outside} \\ -q\rho^2\dot{\theta}B/2 & \text{inside.} \end{cases}$$ [5]

(b) The Lagrangian has cylindrical symmetry and does not depend explicitly on time. Find the corresponding conserved quantities, and (given the above initial conditions) find $\dot{\theta}$ for the particle inside the solenoid. [5]

(c) Find the equations of motion for z and ρ inside and outside the solenoid, and discuss whether $\dot{z}$ is a constant of the motion. [4]

(d) Find the full motion of the particle within the solenoid and draw a diagram of its path. You may assume without proof that ρ and $\dot{\rho}$ are continuous when the particle enters the solenoid. [7]

(e) Neglecting variations in $\dot{z}$ (which is justified if v is large enough) show that, if a non-interacting beam of particles with equal initial velocity passes through the solenoid, it will be focussed to a point on the z axis a distance

$$f = \frac{v}{\omega}\cot\left(\frac{\omega l}{v}\right)$$

beyond the end of the solenoid, and give an expression for ω. Assume $l < v\pi/(2\omega)$. [4]

[*In cylindrical polar coordinates*

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\theta}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\theta)}{\partial \rho} - \frac{\partial A_\rho}{\partial \theta}\right)\hat{\boldsymbol{z}}$$
]

3 The Lagrangian density for a self-interacting, complex scalar field in 3+1 dimensions $\phi(\boldsymbol{r}, t)$ is given by:

$$\mathcal{L} = (\partial_\mu \phi^*)\,(\partial^\mu \phi) - V(\phi)\,,$$

where $\partial_\mu = \partial/\partial x^\mu$ and $V(\phi) = -\phi^*\phi + \exp(\lambda\,\phi^*\phi)$.

(a) Derive an expression for the Hamiltonian density $\mathcal{H}$ in terms of ϕ and its derivatives, and explain why only positive values of λ are acceptable. Discuss which field values minimise the energy as a function of λ. [6]

(b) For $\lambda \in (0, 1)$, the system breaks spontaneously the global phase symmetry by selecting one of the minima of the energy. Consider small fluctuations about the minimum $\phi_0 = \sqrt{-\ln(\lambda)/\lambda}$: $\phi = \phi_0 + \chi$ and $\phi^* = \phi_0 + \chi^*$. Obtain the Lagrangian density of the system to second order in χ, χ^*. Use your results to illustrate the concept of a Goldstone mode. [8]

(c) The complex scalar field can be coupled to an electromagnetic gauge field A^μ by changing the derivative terms to covariant derivatives ($\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$), and by adding the Lagrangian density of free electromagnetic fields:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^*\,(D_\mu \phi) - V(\phi)\,.$$

Discuss briefly what happens to the Goldstone mode in the spontaneously broken symmetry phase of the scalar field, and how the behaviour of the electromagnetic field is changed. You may quote formulae without deriving them. [3]

(d) Finally, consider adding a further term to the potential $V(\phi)$ in the original Lagrangian density, in the absence of electromagnetic fields:

$$V(\phi) = -\frac{1}{2}\left(\phi^{*2} + \phi^2\right) - \phi^*\phi + \exp(\lambda\,\phi^*\phi)\ .$$

What is the symmetry of the system? Find the minima of the potential as a function of $\lambda > 0$. In the phase where the symmetry is spontaneously broken, expand the potential for small fluctuations χ about one of the minima of your choice, to second order. Show that in this case both components of the field χ have positive mass. [8]

[*You may find it useful to minimise $V(\phi = \phi_0 e^{i\theta})$ with respect to $\theta \in [0, 2\pi)$ and $\phi_0 > 0$.*]

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4 A charged particle radiates energy at a rate proportional to the square of its acceleration, $\ddot{x}^2$. For *periodic* motion, this is equivalent to the action of a force $\dddot{x}$ on the particle. If we consider simple harmonic oscillations in one dimension, the equation of motion can then be written as

$$\ddot{x} + x - \varepsilon \dddot{x} = f(t) \qquad\qquad \varepsilon \in \mathbb{R}\,,$$

where $f(t)$ is an external driving force.

(a) Obtain an expression for the Green's function $G(\omega)$ such that the Fourier transform of the solution of the equation of motion can be written as $x(\omega) = G(\omega) f(\omega)$. [4]

Discuss in general what conditions $G(t-t')$ must satisfy in order to respect causality, and how this relates to the position of the poles of $G(\omega)$ in the complex ω plane. [4]

[*Use the Fourier transform convention:*

$$G(t-t') = \int G(\omega)\, e^{-i\omega(t-t')}\, \frac{d\omega}{2\pi}\,.$$

]

(b) Find the two poles $\omega^{(0)}_{1,2}$ for $\varepsilon = 0$ and show how they are affected to leading order in ε by the addition of the term $-\varepsilon\dddot{x}$ in the equation of motion. [7]

Discuss causality in relation to the sign of ε. How does it compare to the case considered in the lecture notes of a damped simple harmonic oscillator $\ddot{x} + x - \varepsilon\dot{x} = f(t)$? [5]

[*You may assume that the poles move, to leading order, by an amount proportional to ε.*]

(c) Show that, to leading order, a third pole of $G(\omega)$ can be found of the form $\omega_3 \simeq \alpha_3/\varepsilon$. Find the value of the constant α_3 and discuss whether this solution is compatible with causality. [5]

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