NATURAL SCIENCES TRIPOS Part II

Wednesday 14 January 2015

THEORETICAL PHYSICS I

Answers

1 (a) Working in (x, y, z) coordinates the position of the mass is

$$\boldsymbol{r} = (a_x, a_y, 0) + l(\sin(\theta), -\cos(\theta), 0)$$

and the velocity is

$$\dot{\boldsymbol{r}} = (\dot{a}_x, \dot{a}_y, 0) + l(\cos(\theta), \sin(\theta), 0)\dot{\theta}$$

so the kinetic and potential energy are:

$$\begin{aligned} T &= \frac{1}{2}m|\dot{\boldsymbol{r}}|^2 = \frac{1}{2}m(\dot{a}_x^2 + \dot{a}_y^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}(\dot{a}_x\cos(\theta) + \dot{a}_y\sin(\theta)) \\ V &= mgz = mg(a_y - l\cos(\theta)). \end{aligned}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{a}_x^2 + \dot{a}_y^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}(\dot{a}_x\cos(\theta) + \dot{a}_y\sin(\theta)) - mg(a_y - l\cos(\theta))$$

(b) We have

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml(\dot{a}_x\cos(\theta) + \dot{a}_y\sin(\theta)), \; \frac{\partial L}{\partial \theta} = ml\dot{\theta}(-\dot{a}_x\sin(\theta) + \dot{a}_y\cos(\theta)) - mgl\sin(\theta).$$

The equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}.$$

The left-hand-side is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(ml^2\dot{\theta} + l(\dot{a}_x\cos(\theta) + \dot{a}_y\sin(\theta))\Big) \\ &= ml^2\ddot{\theta} + ml(\ddot{a}_x\cos(\theta) + \ddot{a}_y\sin(\theta) + \dot{\theta}(\dot{a}_y\cos(\theta) - \dot{a}_x\sin(\theta))), \end{aligned}$$

while the left hand side is given above. Putting the two together, the $\dot{\theta}$ terms cancel, giving the equation of motion

$$\begin{aligned} l^2\ddot{\theta} + l(\ddot{a}_x\cos(\theta) + \ddot{a}_y\sin(\theta)) &= -gl\sin(\theta) \\ \implies \quad l^2\ddot{\theta} &= -l[(\ddot{a}_y + g)\sin(\theta) + \ddot{a}_x\cos(\theta)] \end{aligned}$$

Writing $\boldsymbol{l} = l(\sin(\theta), -\cos(\theta), 0)$, $\boldsymbol{g} = (0, -g, 0)$ and $\ddot{\boldsymbol{a}} = (\ddot{a}_x, \ddot{a}_y, 0)$, we have

$$\boldsymbol{l} \times (\boldsymbol{g} - \ddot{\boldsymbol{a}}) = -l(0, 0, \sin(\theta)(\ddot{a}_y + g) + \cos(\theta)\ddot{a}_x),$$

so, as stated, we can write the equation of motion as:

$$l^2\ddot{\theta}\hat{\boldsymbol{z}} = \boldsymbol{l} \times (\boldsymbol{g} - \ddot{\boldsymbol{a}})$$

(c) We now have $a_y = 0$, $a_x = a\cos(\omega t)$, so our equation of motion is

$$l\ddot{\theta} = -g\sin(\theta) + a\omega^2\cos(\omega t)\cos(\theta).$$

However, we are told that

$$\theta = \theta_1 - \frac{a}{l}\cos(\theta_1)\cos(\omega t).$$

where θ_1 is a slowly varying function of time and $a/l \ll 1$. We insert this into the above equation, expand to first order in a/l and average over one cycle, $2\pi/\omega$, in which time θ_1 does not significantly change. The lhs simply gives $l\ddot{\theta}_1$ since all the other terms time-average to zero. On the right hand side, we have

$$-g\sin(\theta) \approx -g\left(\sin(\theta_1) - \cos(\theta_1)\frac{a}{l}\cos(\theta_1)\cos(\omega t)\right),$$

but, averaging over one cycle the second term on the right gives zero, so this becomes

$$-g\sin(\theta) \to -g\sin(\theta_1).$$

The other term on the right hand side is

$$a\omega^2\cos(\omega t)\cos(\theta) \approx a\omega^2\cos(\omega t)\left(\cos(\theta_1) + \sin(\theta_1)\frac{a}{l}\cos(\omega t)\cos(\theta_1)\right).$$

Time averaging, the first term gives zero while $\cos^2(\omega t) \to 1/2$, so this becomes

$$a\omega^2\cos(\omega t)\cos(\theta) \to \frac{1}{2}\frac{a^2}{l}\omega^2\sin(\theta_1)\cos(\theta_1).$$

Assembling these three pieces, the equation for θ_1 is

$$l\ddot{\theta}_1 = -g\sin(\theta_1) + \frac{1}{2}\frac{a^2}{l}\omega^2\sin(\theta_1)\cos(\theta_1).$$

(d) If this were an un-driven pendulum but in a different potential $V_{\text{eff}}(\theta)$, the equation of motion would be

$$ml^2\ddot{\theta} = -V'_{\text{eff}}(\theta).$$

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For this to match the motion of θ_1, we need

$$\begin{aligned} V_{\text{eff}}(\theta) &= ml \int \left(g\sin(\theta) - \frac{1}{2}\frac{a^2}{l}\omega^2 \sin(\theta)\cos(\theta) \right) \mathrm{d}\theta \\ &= ml \left(-g\cos(\theta) + \frac{a^2\omega^2}{4l}\cos^2(\theta) \right) \end{aligned}$$

For small a the first term, $\propto \cos(\theta)$, dominates so the potential has a single minimum at $\theta = 0$. At large a the second term dominates, so the potential has a maxima when $\theta = 0$ and minima at $\pm\pi/2$. The transition happens when $V''_{eff}(0) = 0$. We have

$$V''_{eff}(\theta) = ml \left(g\cos(\theta) - \frac{a^2\omega^2}{2l}(\cos^2(\theta) - \sin^2(\theta)) \right),$$

so $V''_{eff}(0) = 0$ when $\frac{1}{2}\frac{a^2\omega^2}{l} = g$, so the $\theta = 0$ solution is unstable if $a^2 > 2lg/\omega^2$.

2 (a) The kinetic energy for the system is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2).$$

The elastic potential energy for the circular spring is

$$V_{spring} = \frac{1}{2}k(2\pi r - 2\pi r_0)^2 = 2\pi^2 k(r - r_0)^2.$$

A convenient form for the magnetic vector potential is $\boldsymbol{A} = B\frac{r}{2}\hat{\boldsymbol{\phi}}$, so $\boldsymbol{v} \cdot \boldsymbol{A} = \frac{1}{2}Br^2\dot{\phi}$ at every point on the ring. The velocity dependent magnetic potential energy is thus

$$V_{mag} = -\frac{qBr^2}{2}\dot{\phi}.$$

Putting these together, the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - 2\pi^2 k(r - r_0)^2 + \frac{qBr^2}{2}\dot{\phi}.$$

(b) The momenta conjugate to r and ϕ are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + \frac{qBr^2}{2} \end{aligned}$$

The Hamiltonian is thus

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + 2\pi^2 k(r - r_0)^2.$$

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However, we should write this in terms of the canonical variables, p_r, p_ϕ, r and ϕ, to get

$$H = \frac{p_r^2}{2m} + \frac{\left(p_\phi - \frac{qBr^2}{2}\right)^2}{2mr^2} + 2\pi^2 k(r - r_0)^2,$$

which is why Hamilton's equations will depend on B.

(c) The system has two symmetries: it does not depend on time or on the coordinate ϕ. The former guarantees that the Hamiltonian will be a constant of the motion, i.e. that the sum of the kinetic and spring energy will be conserved during motion. The latter guarantees, via Lagrange's equation of motion for ϕ, that p_ϕ is conserved during motion.

Lagrange's equation of motion for r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$
$$\implies \quad m\ddot{r} = mr\dot{\phi}^2 + qBr\dot{\phi} - 4\pi^2 k(r - r_0)$$

However, we know $p_\phi = mr^2\dot{\phi} + \frac{qBr^2}{2} \equiv J$ is a constant of the motion, so we use it to eliminate $\dot{\phi}$.

$$m\ddot{r} = mr\left(\frac{J}{mr^2} - \frac{qB}{2m}\right)^2 + qBr\left(\frac{J}{mr^2} - \frac{qB}{2m}\right) - 4\pi^2 k(r - r_0)$$
$$\implies \quad m\ddot{r} = \frac{J^2}{mr^3} - \frac{q^2B^2r}{4m} - 4\pi^2 k(r - r_0)$$

(d) With the given value of $\dot{\phi}$, we see that $J = 0$. The equation of motion for r thus reduces to

$$m\ddot{r} = -\frac{q^2B^2r}{4m} - 4\pi^2 k(r - r_0)$$

This is just an SHM equation for r. The equilibrium radius, r_e, is where the rhs=0, i.e. where

$$-\frac{q^2B^2r_e}{4m} - 4\pi^2 k(r_e - r_0) = 0 \implies r_e = \frac{4\pi^2 kr_0}{\frac{q^2B^2}{4m} + 4\pi^2 k} = \frac{16\pi^2 km}{B^2q^2 + 16\pi^2 km} r_0.$$

We can then write the equation of motion in the form

$$m\ddot{r} = -\left(\frac{q^2B^2}{4m} + 4\pi^2 k\right)(r - r_e)$$

So the solution we are looking for is

$$r = r_e + (r_0 - r_e)\cos(\omega t)$$

with r_e given above and $m\omega^2 = \left(\frac{q^2B^2}{4m} + 4\pi^2 k\right)$. Since $J = 0$ is a constant of the motion,we have, at all times, $\dot{\phi} = -\frac{qB}{2m}$, so the solution for ϕ is simply

$$\phi = -\frac{qB}{2m}t.$$

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(e) In this motion, we start with $\dot{r} = \dot{\phi} = 0$ and $r = r_1$, so we have $p_\phi = mr^2\dot{\phi} + \frac{qBr^2}{2} = \frac{qBr_1^2}{2}$, which we can solve to find $\dot{\phi}$ as:

$$\dot{\phi} = \frac{qB}{2m}\left(\frac{r_1^2}{r^2} - 1\right).$$

We can also evaluate the Hamiltonian as

$$H = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + 2\pi^2 kr^2 = 2\pi^2 kr_1^2.$$

Substituting our equation for ϕ into the Hamiltonian gives

$$H = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\left(\frac{qB}{2m}\right)^2\left(\frac{r_1^2}{r^2} - 1\right)^2 + 2\pi^2 kr^2 = 2\pi^2 kr_1^2.$$

At the minimum radius we must have $\dot{r} = 0$, yielding the following quartic equation for r,

$$\begin{aligned} &\tfrac{1}{2}mr^2\left(\tfrac{qB}{2m}\right)^2\left(\tfrac{r_1^2}{r^2} - 1\right)^2 + 2\pi^2 kr^2 = 2\pi^2 kr_1^2 \\ \Longrightarrow\quad &\tfrac{q^2B^2}{8m}\left(r_1^4 - 2r_1^2r^2 + r^4\right) + 2\pi^2 k(r^4 - r_1^2r^2) = 0 \\ \Longrightarrow\quad &\left(\tfrac{q^2B^2}{8m} + 2\pi^2 k\right)r^4 - \left(\tfrac{q^2B^2}{4m} + 2\pi^2 k\right)r_1^2r^2 + \tfrac{q^2B^2}{8m}r_1^4 = 0 \end{aligned}$$

This equation has four roots,

$$r = \pm r_1 \qquad r = \pm\frac{Bq}{\sqrt{B^2q^2 + 16km\pi^2}}r_1,$$

The minimum radius of the motion is given by the positive root less than r_1:

$$r = \frac{Bq}{\sqrt{B^2q^2 + 16km\pi^2}}r_1.$$

At this minimum radius the angular velocity is given by

$$\dot{\phi} = \frac{qB}{2m}\left(\frac{r_1^2}{r^2} - 1\right) = \frac{8\pi^2 k}{Bq}.$$

3 (a) The Poisson bracket of two functions f and g is defined as

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

A change of variables $q, p \to Q(q,p), P(q,p)$ is said to be a canonical transformation if it satisfies the condition

$$\{Q, P\} = 1 \qquad \text{(and trivially: } \{Q, Q\} = 0 = \{P, P\}\text{)}$$

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where the Poisson brackets are defined with respect to the original coordinates and momenta q, p. Under canonical transformations, Hamilton's equations of motion are preserved; namely, upon transforming the Hamiltonian, one can use Hamilton's equations with respect to Q, P to obtain the equations of motion of the system in the new variables.

One can show that the total time derivative of a generic observable $\mathcal{O}(q,p,t)$ of the system can be obtained using the Poisson bracket of the function with the Hamiltonian:

$$\frac{d\mathcal{O}}{dt} = \{\mathcal{O}, H\} + \frac{\partial \mathcal{O}}{\partial t}.$$

(b) Hamilton's equations of motion are:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = \cosh(2q) \\ \dot{p} = -\frac{\partial H}{\partial q} = -2p\sinh(2q) \end{cases}$$

The function $\mathcal{O}(q,p)$ does not depend explicitly on time; therefore

$$\frac{d\mathcal{O}}{dt} = \{\mathcal{O}, H\} = p\, e^{qp}\frac{\partial H}{\partial p} - q\, e^{qp}\frac{\partial H}{\partial q} = (p\,\dot{q} + q\,\dot{p})\, e^{qp}.$$

(c) In order for this change of variables to be a canonical transformation, it must satisfy the condition

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = f(p)f'(p)\cosh^2(q) - f'(p)f(p)\sinh^2(q) = f(p)f'(p) = 1.$$

One can verify that $f(p) = \sqrt{2p}$ satisfies this equation.

Using the identity $\cosh(2x) = \sinh^2(x) + \cosh^2(x)$, we can re-write the Hamiltonian as

$$H = p\left[\sinh^2(q) + \cosh^2(q)\right] = \frac{1}{2}\left[\sqrt{2p}\sinh(q)\right]^2 + \frac{1}{2}\left[\sqrt{2p}\cosh(q)\right]^2$$

which, after the chosen canonical change of variables, reduces to $H = Q^2/2 + P^2/2$, the Hamiltonian of a simple harmonic oscillator (of mass and elastic constant $m = k = 1$).

(d) Hamilton's equations of motion applied to $H - g(t)Q$ are trivially $\dot{Q} = \partial H/\partial P = P$ and $\dot{P} = -\partial H/\partial Q = -Q + g(t)$. Taking the time derivative of the first equation and substituting the second one into it, we arrive at the result:

$$\ddot{Q} + Q = g(t) = \frac{\omega_0}{2}e^{-\omega_0|t|}.$$

The Green's function is a solution of the same equation with a δ-source, $\ddot{\mathcal{G}}(t-t') + \mathcal{G}(t-t') = \delta(t-t')$. Using the Fourier transform $\mathcal{G}(t-t') = \int d\omega\, G(\omega)\exp[-i\omega(t-t')]/2\pi$, the Green's function can be computed

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explicitly as $G(\omega) = (1-\omega^2)^{-1}$. This function has two poles on the real axis at $\omega = \pm 1$. When computing the real time Green's function $\mathcal{G}(t-t')$ one has to move the poles down in the negative imaginary half plane in order to satisfy the physical requirement of causality, namely $\mathcal{G}(t-t') = 0$ for $t < t'$.

Rather than transforming the Green's function back to real time, we follow the suggestion in the hint and compute firstly $Q(\omega) = G(\omega)g(\omega)$ in Fourier space. We must then evaluate the integral

$$Q(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\exp[-i\omega t]}{1-\omega^2} \frac{\omega_0^2}{\omega^2+\omega_0^2}$$

The integrand has two poles along the real axis at $\omega = \pm 1$ – which we push into the negative imaginary half plane to satisfy causality – and two poles on the imaginary axis at $\omega = \pm i\omega_0$.

In order to solve the integral, we use Cauchy's theorem. For $t < 0$ we must close the contour along an infinite semicircle in the upper half plane. This encircles only one pole at $\omega = i\omega_0$ (recall that $\omega_0 > 0$) in the counter-clockwise direction, and therefore:

$$Q(t<0) = \frac{2\pi i}{2\pi} \frac{\exp(\omega_0 t)}{1+\omega_0^2} \frac{\omega_0^2}{2i\omega_0} = \frac{\exp(-\omega_0|t|)}{2} \frac{\omega_0}{1+\omega_0^2}.$$

For $t > 0$ we must close the contour along an infinite semicircle in the lower half plane. This encircles the three remaining poles, at $\omega = -i\omega_0$ and $\omega = \pm 1$, in the clockwise direction. Therefore:

$$Q(t>0) = -\frac{2\pi i}{2\pi} \left\{ \frac{\exp(-it)}{-2} \frac{\omega_0^2}{1+\omega_0^2} + \frac{\exp(it)}{2} \frac{\omega_0^2}{1+\omega_0^2} + \frac{\exp(-\omega_0 t)}{1+\omega_0^2} \frac{\omega_0^2}{-2i\omega_0} \right\}$$

which simplifies to

$$Q(t>0) = \frac{\omega_0^2}{1+\omega_0^2}\sin(t) + \frac{\exp(-\omega_0 t)}{1+\omega_0^2}\frac{\omega_0}{2} = \frac{\omega_0}{1+\omega_0^2}\left\{\omega_0 \sin(t) + \frac{\exp(-\omega_0 t)}{2}\right\}$$

The behaviour of $Q(t)$ is continuous at $t = 0$ but singular in its first derivative (not surprisingly, since $g(t)$ is singular at $t = 0$). It is the superposition of a symmetric contribution $\propto \exp(-\omega_0|t|)$ and a contribution $\propto \sin(t)$ that is present only at positive times due to our choice of imposing causality on the Green's function.

4 (a) The velocity of an element $\mathrm{d}x$ of the string is $\dot{\psi}_y\hat{\boldsymbol{y}} + \dot{\psi}_z\hat{\boldsymbol{z}}$, so the total kinetic energy is

$$T = \int \frac{1}{2}\rho(\dot{\psi}_y^2 + \dot{\psi}_z^2)\mathrm{d}x.$$

The elastic potential energy is

$$V = F\left[\int \mathrm{d}s - \int \mathrm{d}x\right] = F\left(\int \sqrt{1+\psi_z'^2+\psi_y'^2}\mathrm{d}x - \int \mathrm{d}x\right)$$

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For small displacement gradients, we can expand this to leading order to get

$$V = \int \frac{1}{2} F \left(\psi_z'^2 + \psi_y'^2\right) \mathrm{d}x.$$

The Lagrangian for the string is thus

$$L = T - V = \int \frac{1}{2}\rho(\dot{\psi}_y^2 + \dot{\psi}_z^2) - \frac{1}{2} F \left(\psi_z'^2 + \psi_y'^2\right) \mathrm{d}x.$$

(b) The transformation $\psi_y \to \psi_y + a$ does not change ψ_y' or $\dot{\psi}_y$ so, trivially, the Lagrangian is invariant. The resultant Noether conservation law is simply the Lagrangian equation of motion for ψ_y:

$$\partial_t \frac{\partial L}{\partial \dot{\psi}_y} + \partial_x \frac{\partial L}{\partial \psi_y'} = \frac{\partial L}{\partial \psi_y} = 0.$$

Evaluating this for the Lagrangian in question gives:

$$\partial_t(\rho\dot{\psi}_y) - \partial_x(F\psi_y') = 0.$$

Integrating along the entire length of the string, we have

$$\int_{-\infty}^{\infty} \partial_t(\rho\dot{\psi}_y)\mathrm{d}x = \int_{-\infty}^{\infty} \partial_x(F\psi_y')\mathrm{d}x = \left[F\psi_y'\right]_{-\infty}^{\infty} = 0.$$

On the left-hand side, we can bring the ∂_t outside the integral, so we have

$$\partial_t \int_{-\infty}^{\infty} (\rho\dot{\psi}_y)\mathrm{d}x = 0$$

i.e. the quantity

$$P_y = \int_{-\infty}^{\infty} (\rho\dot{\psi}_y)\mathrm{d}x,$$

which is the total linear momentum in the y direction, is conserved in time.

(c) A global rotation of the displacement fields is supplied by the transformation $\psi_y \to \psi_y \cos(\theta) - \psi_z \sin(\theta)$, $\psi_z \to \psi_z \cos(\theta) + \psi_y \sin(\theta)$. We therefore have

$$\begin{aligned} \dot{\psi}_y^2 + \dot{\psi}_z^2 &\to \left(\dot{\psi}_y \cos(\theta) - \dot{\psi}_z \sin(\theta)\right)^2 + \left(\dot{\psi}_z \cos(\theta) + \dot{\psi}_y \sin(\theta)\right)^2 \\ &= \dot{\psi}_y^2 + \dot{\psi}_z^2 \end{aligned}$$

and, similarly,

$$\begin{aligned} \psi_y'^2 + \psi_z'^2 &\to \left(\psi_y' \cos(\theta) - \psi_z' \sin(\theta)\right)^2 + \left(\psi_z' \cos(\theta) + \psi_y' \sin(\theta)\right)^2 \\ &= \psi_y'^2 + \psi_z'^2, \end{aligned}$$

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so the Lagrangian is indeed invariant.

In this case, the conserved quantity does not appear directly from equations of motion. Instead, we consider the infinitesimal form of the transformation, $\psi_y \to \psi_y + \delta\psi_y$, $\psi_z \to \psi_z + \delta\psi_z$, with

$$\delta\psi_y = -\psi_z\theta \qquad \delta\psi_z = \psi_y\theta.$$

The conserved density is then

$$\begin{aligned} P_\theta &= \int \left(\tfrac{\partial L}{\partial \dot{\psi}_y}\delta\psi_y + \tfrac{\partial L}{\partial \dot{\psi}_z}\delta\psi_z \right) \mathrm{d}x \\ &= \int \rho \left(-\dot{\psi}_y\psi_z + \dot{\psi}_z\psi_y \right) \mathrm{d}x, \end{aligned}$$

which we recognize as the total angular momentum of the string around the x axis.

(d) The Lagrangian is now

$$L = T - V = \int \frac{1}{2}\rho\dot{\psi}_y^2 - \frac{1}{2}F\psi_y'^2 - \frac{1}{2}B\psi_y''^2 \mathrm{d}x.$$

The Euler-Lagrange equation for the rod is

$$\partial_t \frac{\partial L}{\partial \dot{\psi}_y} + \partial_x \frac{\partial L}{\partial \psi_y'} - \partial_x\partial_x \frac{\partial L}{\partial \psi_y''} = 0,$$

which we can evaluate to get

$$\rho\ddot{\psi}_y - F\psi_y'' + B\psi_y'''' = 0.$$

We try a wave solution for the form $\psi_y = \mathrm{e}^{\mathrm{i}(kx-\omega t)}$. Putting this into the Euler-Lagrange equation, we get

$$(-\rho\omega^2 + Fk^2 + Bk^4)\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0,$$

which is true provided we obey the dispersion relation

$$\omega^2 = \frac{k^2}{\rho}(F + Bk^2).$$

(e) Recalling F is negative (i.e. the rod is in compression), the rhs of the dispersion relation is negative if

$$k^2 < -\frac{F}{B},$$

i.e. if the wavelength is sufficiently long. If the rhs is negative, then ω is imaginary, so we can write $\omega = \pm\mathrm{i}/\tau$, where

$$\frac{1}{\tau} = \sqrt{-\frac{k^2}{\rho}(F + Bk^2)},$$

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is real. Our wave solution is then

$$e^{\mathrm{i}(kx-\omega t)} = e^{\mathrm{i}(kx\mp \mathrm{i}t/\tau)} = e^{\mathrm{i}kx}e^{\pm t/\tau},$$

and one of these solutions grows exponentially in time, indicating that the straight compressed rod is unstable.

The ends of the rod have zero displacement, so the solution of this type with the longest admissible wavelength has $\lambda = 2l \implies k = \pi/l$. Such a wavelength will indeed be unstable if

$$\frac{\pi^2}{l^2} < -\frac{F}{B} \qquad \Rightarrow \qquad l > \pi\sqrt{\frac{-B}{F}}.$$

5 (a) The Klein-Gordon Lagrangian density for a complex scalar field

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi^*\phi$$

can be coupled to electromagnetic fields by replacing the derivatives with so-called *covariant derivatives* $D_\mu = \partial_\mu + ieA_\mu$:

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* - m^2\phi^*\phi.$$

Upon expanding the covariant derivatives, one can show that this coupling amounts to the standard $J^\mu_{KG}A_\mu$ coupling, plus additional terms.

This choice of coupling to electromagnetism via the covariant derivative is special in that it realises a Lagrangian density that is invariant under local phase transformations $\phi \to e^{-i\epsilon(x)}\phi$, $\phi^* \to e^{i\epsilon(x)}\phi^*$. Indeed, if we accompany such change with a simultaneous gauge transformation $A_\mu \to A_\mu + \partial_\mu\epsilon(x)/e$, one can verify that the covariant derivatives transform just like the fields themselves: $D_\mu\phi \to e^{-i\epsilon}D_\mu\phi$ and $(D_\mu\phi)^* \to e^{i\epsilon}(D_\mu\phi)^*$. Therefore, the Lagrangiam remains unchanged.

(b) Nöther's theorem states that there is a conserved current associated with every continuous symmetry of the Lagrangian, i.e. with symmetry under a transformation of the form $\phi \to \phi + \delta\phi$, where $\delta\phi$ is infinitesimal. The conserved current takes the form

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi$$

and the conserved charge is the spatial integral of the time component of the conserved current, $Q = \int d^3r J^0(\boldsymbol{r}, t)$.

(c) The Euler-Lagrange equations can be obtained from

$$\begin{cases} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi} \\ \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \frac{\partial\mathcal{L}}{\partial\phi^*} \end{cases}$$

After a few lines of algebra, we obtain

$$\begin{cases} \partial_\mu\partial^\mu\phi^* - ie\partial_\mu(A^\mu\phi^*) = -m^2\phi^* + ieA_\mu\partial^\mu\phi^* \\ \partial_\mu\partial^\mu\phi + ie\partial_\mu(A^\mu\phi) = -m^2\phi - ieA_\mu\partial^\mu\phi \end{cases}$$

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or equivalently

$$\begin{cases} \partial_\mu \partial^\mu \phi^* - ie\phi^* \partial_\mu A^\mu - 2ieA^\mu \partial_\mu \phi^* + m^2 \phi^* = 0 \\ \partial_\mu \partial^\mu \phi + ie\phi \partial_\mu A^\mu + 2ieA^\mu \partial_\mu \phi + m^2 \phi = 0 \end{cases} .$$

In order to obtain the Hamiltonian density, we need first to compute the canonical momentum densities

$$\begin{aligned} \pi(\boldsymbol{r}, t) &= \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial^t \phi^* - ieA^0 \phi^* \\ \pi^*(\boldsymbol{r}, t) &= \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = \partial^t \phi + ieA^0 \phi \end{aligned}$$

and then construct

$$\mathcal{H} = \pi \partial_t \phi + \pi^* \partial_t \phi^* - \mathcal{L} .$$

Since the question asks explicily for a Hamiltonian density expressed as a function of its proper variables (namely, ϕ, π, $\nabla\phi$ and their complex conjugates), we invert the equations for the canonical momentum densities

$$\begin{aligned} \partial^t \phi^* &= \pi(\boldsymbol{r}, t) + ieA^0 \phi^* \\ \partial^t \phi &= \pi^*(\boldsymbol{r}, t) - ieA^0 \phi \end{aligned}$$

and substitute into $\mathcal{H}$ to obtain

$$\begin{aligned} \mathcal{H} &= \pi(\pi^* - ieA_0\phi) + \pi^*(\pi + ieA_0\phi^*) \\ &\quad - (\pi + ieA_0\phi^*)\left(\pi^* - ieA^0\phi\right) - (\partial_i\phi^*)\left(\partial^i\phi\right) + m^2\phi^*\phi \\ &\quad -ieA_0\left[\phi(\pi + ieA^0\phi^*) - \phi^*(\pi^* - ieA^0\phi)\right] - ieA_i\left[\phi\partial^i\phi^* - \phi^*\partial^i\phi\right] \\ &= \pi\pi^* - ieA_0\phi\pi + ieA_0\phi^*\pi^* \\ &\quad +ieA^0\phi\pi - ieA_0\phi^*\pi^* - e^2A_0A^0\phi^*\phi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi \\ &\quad -ieA_0\left[\phi\pi - \phi^*\pi^*\right] - ieA_i\left[\phi\partial^i\phi^* - \phi^*\partial^i\phi\right] \\ &= \pi\pi^* + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi - e^2A_0A^0\phi^*\phi - ieA_0\left[\phi\pi - \phi^*\pi^*\right] - ie\boldsymbol{A}\cdot\left[\phi\nabla\phi^* - \phi^*\nabla\phi\right] \end{aligned}$$

where we have introduced $\boldsymbol{A} \equiv (A^1, A^2, A^3)$.

(d) Nöther's current for the Lagrangian density in part (c) can be obtained from

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^*$$

with respect to an infinitesimal global phase change $\phi \to \phi - i\epsilon\phi$, $\phi^* \to \phi^* + i\epsilon\phi^*$.

Neglecting the constant overall factor of ϵ, we arrive at:

$$J^\mu = -i\phi\left[\partial^\mu\phi^* - ieA^\mu\phi^*\right] + i\phi^*\left[\partial^\mu\phi + ieA^\mu\phi\right] = -i\left[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right] - 2eA^\mu\phi^*\phi.$$

We can explicitly check that Noether's current is conserved by computing

$$\partial_\mu J^\mu = -i\left[\phi\partial_\mu\partial^\mu\phi^* - \phi^*\partial_\mu\partial^\mu\phi\right] - 2e(\partial_\mu A^\mu)\phi^*\phi - 2eA^\mu\left[\phi^*\partial_\mu\phi + \phi\partial_\mu\phi^*\right]$$

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and using the Euler-Lagrange equations derived earlier to substitute:

$$\begin{array}{l} \partial_\mu \partial^\mu \phi^* = ie\phi^* \partial_\mu A^\mu + 2ieA^\mu \partial_\mu \phi^* - m^2 \phi^* \\ \partial_\mu \partial^\mu \phi = -ie\phi \partial_\mu A^\mu - 2ieA^\mu \partial_\mu \phi - m^2 \phi \end{array} .$$

This gives

$$\partial_\mu J^\mu = [2e\phi^* \phi \partial_\mu A^\mu + 2eA^\mu (\phi \partial_\mu \phi^* + \phi^* \partial_\mu \phi)] - 2e(\partial_\mu A^\mu)\phi^* \phi - 2eA^\mu [\phi^* \partial_\mu \phi + \phi \partial_\mu \phi^*]$$

which vanishes identically as the terms cancel out pairwise.

6 (a) MFT approximates the interaction of a spin with its neighbours by assuming that its neighbours behave as a typical spin in the system. Namely, each spin interacts with the thermodynamic average of its neighbours. It becomes increasingly more reliable as the number of neighbours increases; it is exact in the limit of infinite dimensions or infinite range interactions.

(b) In MFT, the energy is approximated by:

$$E = -\frac{J}{2} \sum_{i,\delta} [\langle \boldsymbol{S}_i \rangle \cdot \boldsymbol{S}_{i+\delta} + \boldsymbol{S}_i \cdot \langle \boldsymbol{S}_{i+\delta} \rangle] = -4J\boldsymbol{S} \cdot \left(\sum_i \boldsymbol{S}_i \right) .$$

The partition function can therefore be written as:

$$Z = \sum_{\{\boldsymbol{S}_i\}} \exp\left[\beta 2dJ\boldsymbol{S} \cdot \left(\sum_i \boldsymbol{S}_i \right) \right] = \prod_i \left[\sum_{\boldsymbol{S}_i} \exp\left(a\boldsymbol{S} \cdot \boldsymbol{S}_i \right) \right] ,$$

where we have introduced for convenience the parameter $a = \beta 2dJ$ (note that $a > 0$).

As suggested in the question, we choose a reference frame for the spins $\boldsymbol{S}_i$ such that the z-axis is in the direction of $\boldsymbol{S}$, which leads to $\boldsymbol{S} \cdot \boldsymbol{S}_i = S\cos\theta$. Following the hint, the summation over all possible values (i.e., orientations) of the unit vector $\boldsymbol{S}_i$ can be written as an integral over the solid angle element,

$$Z = \left[2\pi \int_0^\pi \sin\theta d\theta \; e^{aS\cos\theta} \right]^N = \left[\frac{4\pi}{aS} \sinh(aS) \right]^N .$$

(c) Following the same steps as in part (b),

$$\begin{aligned} \langle \boldsymbol{S}_k \cdot \hat{z} \rangle &= \frac{1}{Z} \sum_{\{\boldsymbol{S}_i\}} (\boldsymbol{S}_k \cdot \hat{z}) \exp\left[a\boldsymbol{S} \cdot \left(\sum_i \boldsymbol{S}_i \right) \right] \\ &= \frac{1}{Z} \left\{ \sum_{\boldsymbol{S}_k} (\boldsymbol{S}_k \cdot \hat{z}) \exp[a\boldsymbol{S} \cdot \boldsymbol{S}_k] \right\} \left\{ \prod_{i \neq k} \sum_{\boldsymbol{S}_i} \exp[a\boldsymbol{S} \cdot \boldsymbol{S}_i] \right\} \\ &= \frac{1}{Z} \left\{ 2\pi \int_0^\pi \sin\theta d\theta \; \cos\theta \, e^{aS\cos\theta} \right\} \left[\frac{4\pi}{aS} \sinh(aS) \right]^{N-1} . \end{aligned}$$

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[Extra: Using a similar calculation, one can show that the x and y components of the vector $\langle \boldsymbol{S}_k \rangle$ vanish upon integration over the azimuthal angle ϕ, as expected by symmetry.]

The integral for the z component reduces to $2\pi \int_{-1}^{1} dx\, x \exp(aSx)$, which can be calculated by parts and one obtains

$$\langle \boldsymbol{S}_k \cdot \hat{z} \rangle = \frac{\frac{4\pi}{(aS)^2}\left[aS\cosh(aS) - \sinh(aS)\right]}{\frac{4\pi}{aS}\sinh(aS)} = \coth(aS) - \frac{1}{aS}.$$

The self-consistency condition of MFT is obtained by imposing that $\langle \boldsymbol{S}_k \cdot \hat{z} \rangle$ is equal to S in $\boldsymbol{S} = (0, 0, S)$ (recall that we had chosen the reference frame for the spins such that $\boldsymbol{S} \parallel \hat{z}$).

We can therefore rewrite the self-consistency condition as

$$S = \coth(aS) - \frac{1}{aS}.$$

Upon changing variable $S \to x = aS$, this equation becomes

$$\frac{x}{a} = \coth(x) - \frac{1}{x},$$

which we can compare with the formula in the question and we can identify $\tau = (\beta 2dJ)^{-1}$ and $x = \beta 2dJS = 3T_cS/T$.

(d) The function $L(x) = \coth(x) - 1/x$ tends to ± 1 for $x \to \pm\infty$. The diveregence of $\coth(x)$ at $x = 0$ is compensated by the subtraction of $1/x$ and therefore $L(0) = 0$. Indeed, for small x one obtains the following expansion:

$$\begin{aligned} L(x) &= \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x} = \frac{2 + x^2 + \mathcal{O}(x^4)}{2x + x^3/3 + \mathcal{O}(x^5)} - \frac{1}{x} \\ &= \frac{2 + x^2 + \mathcal{O}(x^4)}{2x}\left[1 - \frac{x^2}{6} + \mathcal{O}(x^4)\right] - \frac{1}{x} \\ &= \frac{2 + 2x^2/3 + \mathcal{O}(x^4)}{2x} - \frac{1}{x} = \frac{x}{3} + \mathcal{O}(x^3). \end{aligned}$$

One can further verify that the derivative of $L(x)$ is always positive. The shape of $L(x)$ is similar to that of $\tanh(x)$ except that the derivative at $x = 0$ takes the value $1/3$ instead of 1. The graphic solution of the equation $\tau x = L(x)$ therefore operates in a similar way as to the case of the Ising model considered in the lecture notes. If the slope of the left hand side is larger than $1/3$, then the only solution is at $x = 0$. This corresponds to $T > T_c$; from $\tau = k_BT/2dJ > 1/3$ we obtain $T_c = 2dJ/3k_B$. If the slope is less than $1/3$ ($T < T_c$), then there are three solutions, at $x = 0$ and at $x = \pm x_0$. As $T \to T_c^-$ the two solutions at $\pm x_0$ tend to 0, where they merge at $T = T_c$.

This behaviour signals (within MFT) that the model undergoes a continuous phase transition at $T = T_c$. Above the transition, the magnetisation of the system

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$M \propto S \propto x$ vanishes as expected in a system that is symmetric upon global rotations of the spins $\boldsymbol{S}_i$. Below the transition the system develops two non-vanishing expectation values of $S \propto \pm x_0$ (one can verify that they are indeed the two minima of the free energy, whereas $S = 0$ is now a local maximum – however, this is not asked explicitly in the question). In choosing one of the two solutions, the system spontaneously breaks the rotational symmetry and develops long range order. The transition is second order, in that the discontinuity is in the derivative of M with respect to T, and therefore in the second derivative of the free energy.

(e) Given $c > 0$, the last term is minimised by a constant $\boldsymbol{m}(x) = \boldsymbol{m}_0$. With this choice, the free energy reduces to

$$f = f_0 + a(T - T_c)\, m_0^2 + b\, m_0^4,$$

where $m_0 = |\boldsymbol{m}_0|$. The extrema are found by setting the derivative with respect to m_0 to 0, which leads to

$$a(T - T_c)\, m_0 + 2b\, m_0^3 = 0 \qquad \rightarrow \qquad m_0 = \begin{cases} 0 & T > T_c \\ \sqrt{\frac{a}{2b}}\sqrt{T_c - T} & T < T_c \end{cases}$$

Consistently with the MFT result above, the transition is second order and it breaks the continuous symmetry in $\boldsymbol{m}$. By developing a free energy minimum at finite $m_0 = |\boldsymbol{m}_0|$, the system is thus forced to choose one of the (infinitely many) minima and therefore select a specific direction $\boldsymbol{m}_0$ in the 3D order parameter space.

END OF PAPER