1 A massless rod of length $l$ makes an angle $\theta(t)$ with the vertical, has a mass $m$ at one end, and is in a constant gravitational field $g = -g\hat{y}$. The other end of the rod is attached with a frictionless hinge to a support, so the rod can rotate in the $xy$-plane. The support itself also moves in the $xy$-plane, having a changing displacement $a(t) = (a_x(t), a_y(t), 0)$ from a fixed origin.

(a) Show that the Lagrangian for this system is: \[ L = \frac{1}{2} m [\dot{a}_x^2 + \dot{a}_y^2 + l^2 \dot{\theta}^2 + 2l \dot{\theta} (\ddot{a}_x \cos \theta + \dot{a}_y \sin \theta)] - mg(a_y - l \cos \theta). \] 

(b) Show that the equation of motion for $\theta$ can be written in the form: \[ l^2 \ddot{\theta} \hat{z} = l \times (g - \ddot{a}), \] where the vector $l$ has length $l$ and points from the hinge to the mass.

(c) The support executes small but rapid horizontal oscillations, $a(t) = a \cos(\omega t)\hat{x}$, with $a/l \ll 1$ and $\omega \gg \sqrt{g/l}$. In this limit the pendulum’s motion is well represented by

\[ \theta = \theta_1 - \frac{a}{l} \cos(\theta_1) \cos(\omega t), \]

(TURN OVER for continuation of question 1)
where $\theta_1$ is a large slowly varying angle and the second term is a small but fast oscillation at $\omega$. Use this separation of time-scales and amplitudes to find an effective equation of motion for $\theta_1(t)$ by expanding the complete equation of motion around $\theta_1(t)$ to first order in $a/l$, and then averaging over the time period of one fast oscillation.

(d) Show that the variation of $\theta_1$ would be reproduced by a simple pendulum (without a moving support) with potential energy given by

$$V_{\text{eff}}(\theta) = ml \left( -g \cos(\theta) + \frac{a^2 \omega^2}{4l} \cos^2(\theta) \right),$$

and hence calculate how large $a$ must be for the $\theta = 0$ configuration to lose stability.
A non-conducting circular elastic ring has natural length $2\pi r_0$, spring constant $k$, mass $m$, and carries an electric charge $q$ which is distributed evenly along its length. The ring is constrained to sit in the $xy$-plane and a perpendicular constant magnetic field $\mathbf{B} = B\hat{z}$ is applied through the ring. The elastic ring spins around its axis (i.e. the $z$ axis) and its radius starts to increase, but the ring always remains circular and centred at the origin.

(a) The elastic ring is spinning at $\dot{\phi}$ and has instantaneous radius $r$. In the gauge where $\mathbf{A} = (Br/2)\hat{\phi}$ and neglecting the electric self-interaction of the ring, show the Lagrangian for this system is:

$$L = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\phi}^2 - 2\pi^2 k (r - r_0)^2 + qBr^2\dot{\phi}. \tag{7}$$

(b) Show that the Hamiltonian for this system has the value

$$H = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\phi}^2 + 2\pi^2 k (r - r_0)^2. \tag{5}$$

Notice that $B$ is absent from this expression. Explain why the magnetic field will nevertheless appear in Hamilton’s equations of motion.

(c) Identify two symmetries of this system (in addition to the gauge symmetry of the magnetic field) and their corresponding conserved quantities. Using one of them, show the Euler-Lagrange equations of motion can be reduced to:

$$m\ddot{r} = \frac{J^2}{mr^3} - \frac{q^2 B^2 r}{4m} - 4\pi^2 k (r - r_0), \tag{8}$$

where $J$ is a constant of the motion.

(d) The elastic ring is released with $r = r_0$, $\phi = 0$, $\dot{\phi} = -qB/(2m)$ and $\dot{r} = 0$. Show that this choice of initial conditions corresponds to $J = 0$ and that the subsequent motion of the radius is of the form

$$r = r_e + (r_0 - r_e) \cos(\omega t).$$

Find expressions for $r_e$ and $\omega$. Find the motion of $\phi$. \hfill \{6\}

(e) The ring is stretched to radius $r = r_1$ and released from rest. If the natural radius of the elastic ring, $r_0$, is negligible, what is the minimum radius the ring will reach in its subsequent motion? What is its radial and angular velocity when it reaches this radius? \hfill \{7\}
(a) Define Poisson brackets and discuss their use in classical mechanics: (i) to define canonical transformations; and (ii) to write the equation of motion of a generic observable \( \mathcal{O}(q, p) \).

(b) Consider the following Hamiltonian of a particle of canonical coordinate \( q \) and momentum \( p \):

\[
H = p \cosh(2q).
\]

Compute Hamilton’s equations of motion. Using the Poisson brackets, derive the equation of motion of \( \mathcal{O}(q, p) = e^{qp} \).

(c) Consider the change of variables

\[
Q = f(p) \sinh(q) \quad \text{and} \quad P = f(p) \cosh(q).
\]

What equation must \( f(p) \) satisfy to be a canonical transformation? With the help of trigonometric properties of the hyperbolic sine and cosine functions, choose \( f(p) \) so that the transformation is canonical and the Hamiltonian can be written as:

\[
H = \frac{Q^2}{2} + \frac{P^2}{2}.
\]

in terms of \( Q \) and \( P \). (You may assume \( p > 0 \) in this part of the question.)

(d) Consider applying a time-dependent perturbation \( g(t) \) conjugate to the dynamical variable \( Q(t) \) by changing the Hamiltonian to \( H - g(t)Q(t) \). Using Hamilton’s equations of motion in the new variables \( Q \) and \( P \) of the perturbed system, obtain the second order differential equation for \( Q \) alone. Then use the Green’s function method to obtain the solution \( Q(t) \) for the specific case where the driving term takes the form

\[
g(t) = \frac{\omega_0}{2} \exp(-\omega_0|t|) \quad (\omega_0 > 0).
\]

Poles on the real axis should be moved according to the principle of causality.

**Hint:** It may be convenient to use the formula:

\[
Q(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega) g(\omega) e^{-i\omega t}
\]

where \( G(\omega) \) is the Fourier transformed Green’s function and \( g(\omega) = \omega_0^2/(\omega^2 + \omega_0^2) \) is the Fourier transform of the driving force above.
A very long elastic string with mass per unit length $\rho$ is aligned along the $x$ axis and an outwards force $F$ pulls at both ends so the string carries a tension $F$.

(a) The string undergoes $y$ and $z$ displacements $\psi_y$ and $\psi_z$. Show that, if the spatial derivatives of these displacements are small, the Lagrangian for the string is

$$L = \int \frac{1}{2} \rho \left[ \left( \frac{\partial \psi_y}{\partial t} \right)^2 + \left( \frac{\partial \psi_z}{\partial t} \right)^2 \right] - \frac{1}{2} F \left[ \left( \frac{\partial \psi_y}{\partial x} \right)^2 + \left( \frac{\partial \psi_z}{\partial x} \right)^2 \right] \, dx.$$  

(b) Show the Lagrangian is invariant under the transformation $\psi_y \rightarrow \psi_y + a$, where $a$ is a constant displacement. Find the corresponding conserved quantity, show that it is conserved, and explain its physical significance.

(c) Show the Lagrangian is also invariant under a global rotation of the string’s displacement around the $x$ axis. Find the corresponding conserved quantity and explain its physical significance.

(d) The string in tension is now replaced by an elastic rod in compression, so $F$ is negative, and we restrict attention to displacements in the $y$ direction, setting $\psi_z$ and all its derivatives to zero. In this case we must add another term to the potential energy,

$$\int \frac{1}{2} B \left( \frac{\partial^2 \psi_y}{\partial x^2} \right)^2 \, dx,$$

that penalizes bending of the rod. Show that waves on the rod will obey the dispersion relation:

$$\omega^2 = \frac{k^2}{\rho} (F + Bk^2).$$  

[Hint: To deal with the second derivative in $L$, repeat the regular derivation of the Euler Lagrange equations but integrate by parts twice.]

(e) If the rod has length $l$ and $\psi_y = 0$ at its ends show, using the above dispersion relation or otherwise, that the straight compressed rod is unstable if $l > \pi \sqrt{-B/F}$.  

(TURN OVER)
(a) In the lecture notes, you have studied a way to couple complex scalar fields to electromagnetism, which leads to a Lagrangian density that is invariant under local phase change. Discuss briefly how this is done by introducing the concept of covariant derivative; write an explicit expression for the resulting Lagrangian and demonstrate that it is invariant under local phase change provided that it is accompanied by an appropriate simultaneous gauge transformation.

(b) Discuss briefly Noether’s theorem and how it relates symmetries and conservation laws. You should include a general form of the conserved current and conserved charge.

(c) Consider the following Lagrangian density:

\[ \mathcal{L} = (\partial_\mu \phi^*) (\partial^\mu \phi) - m^2 \phi^* \phi + i e A_{\mu}(r, t) \left[ \phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi \right]. \]

Find the Euler-Lagrange equations and obtain the Hamiltonian density \( \mathcal{H} \) expressed as a function of its proper variables.

(d) The Lagrangian density \( \mathcal{L} \) in part (c) is invariant under global phase change \( \phi \rightarrow e^{-i\epsilon} \phi \) and \( \phi^* \rightarrow e^{i\epsilon} \phi^* \). Use the result of Noether’s theorem to compute the conserved current \( J^\mu \). Then demonstrate explicitly that \( \partial_\mu J^\mu = 0 \) using the Euler-Lagrange equations obtained earlier.
6 (a) Explain briefly the principles of mean field theory and under what circumstances it is expected to be a reliable approximation.

(b) Consider a large 2D square lattice of spins $S_i$, $i = 1, \ldots, N$, with unit norm $|S_i| = 1$ and energy

$$E = -\frac{J}{2} \sum_{i,\delta} S_i \cdot S_{i+\delta} \quad (J > 0).$$

The summation is over all sites $i$ and their respective four nearest neighbouring sites labelled by $i + \delta$ (ignore boundary effects.). Using mean field theory (MFT), compute the approximate partition function of the system, $Z = \sum \{S_i\} \exp(-\beta E)$. Notice that the average value of a spin is a vector $S = \langle S_i \rangle$ whose norm $S$ can be smaller than 1. It is convenient in the calculations to adopt a reference frame for the spins $S_i$ such that $S$ points along the z axis.

(c) Within MFT, compute the expectation value of the z-component of a spin

$$\langle S_k \cdot \hat{z} \rangle = \frac{1}{Z} \sum_{\{S_i\}} (S_k \cdot \hat{z}) \exp(-\beta E)$$

and derive from it the self-consistency condition for the z-component of the average spin $S$. Show that it can be expressed in the form $\tau x = \coth(x) - 1/x$, where $x \propto S$. Find the expressions for $x$ and for the proportionality constant $\tau$.

(d) Discuss how the (graphical) solutions to the self-consistency equation vary as a function of $\tau$ and find the transition temperature $T_c$ in terms of the parameters of the system. Comment on the physical significance of the result. [Hint: you should think carefully about the behaviour of the function $\coth(x) - 1/x$ near the origin.]

(e) The order parameter for the model discussed here is a 3-component vector $m(x)$. Assume the following free energy:

$$f = f_0 + a(T - T_c) m(x) \cdot m(x) + b |m(x) \cdot m(x)|^2 + c |\nabla m(x)| \cdot |\nabla m(x)|,$$

with $a, b, c$ positive constants. Find the value(s) of the field $m(x)$ that minimise the free energy as a function of $T$. What type of transition takes place at $T = T_c$? What symmetry is spontaneously broken?

END OF PAPER