1 Consider a double pendulum composed of two masses \(m_1\) and \(m_2\) attached to two rigid massless rods of equal length \(\ell\), as illustrated in the figure. The two rods are connected by a frictionless hinge at point \(B\) and the other end of the first rod is pinned by a frictionless hinge to rotate about point \(A\). A massless spring of elastic constant \(\kappa\) connects the end points \(A\) and \(C\).

(a) Consider the case \(m_1 = m_2 = m\). Derive the Lagrangian of the system as a function of the angles \(\theta\) and \(\phi\). Expand it to second order assuming that both angles as well as their time derivatives are small. Show that the result can be written as

\[
L = m\ell^2 \left( \dot{\theta}^2 + \frac{1}{2} \dot{\phi}^2 \right) - mg\ell \left( \theta^2 + \frac{1}{2} \phi^2 \right) + \frac{1}{2} \kappa \ell^2 (\theta - \phi)^2,
\]

up to irrelevant constants. [9]

(TURN OVER for continuation of question 1)
(b) From the Euler-Lagrange equations, derive the equations of motion. For what value(s) of the parameters is there a solution where both $\theta$ and $\phi$ oscillate with the same frequency, and satisfy the initial conditions $\theta(0) = -\phi(0) = \xi, \dot{\theta}(0) = \dot{\phi}(0) = 0$? [Note: the generic solution is much more involved!] Describe in words the resulting motion of the pendulum. [7]

(c) Obtain the Lagrangian for the case $m_1 = 0, m_2 = m$. Show that the Euler-Lagrange equations of motion in this case can be written in terms of the variables $\eta = \theta + \phi$ and $\nu = \theta - \phi$,

$$\begin{cases}
\ddot{\eta} + \omega_0^2 \eta = 0 \\
(\omega_0^2 - \omega_1^2) \nu = 0
\end{cases},$$

where $\omega_0^2 = g/2\ell$ and $\omega_1^2 = \kappa/m$. Comment briefly on the nature of the resulting motion and what happens if $\omega_0 = \omega_1$. [5]

(d) We now add a friction term to the equations of motion in case (c) above, $\gamma \theta + \gamma \phi = \gamma \dot{\eta}, \gamma > 0$, and a time-dependent external force $\exp(-\alpha t), \alpha > 0$, that couples only to the sum of the two angles for $t > 0$:

$$\ddot{\eta} + \gamma \dot{\eta} + \omega_0^2 \eta = \begin{cases}
0 & t < 0 \\
A e^{-\alpha t} & t \geq 0
\end{cases},$$

where $A$ is a constant of dimensions $(\text{time})^{-2}$.

Find the Green’s function for $\eta(t)$ by solving the equation

$$\ddot{\eta} + \gamma \dot{\eta} + \omega_0^2 \eta = \delta(t - t')$$

via the Fourier transform

$$\hat{\eta}(\omega) = \int dt e^{-i\omega t} \eta(t), \quad \eta(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \hat{\eta}(\omega),$$

assuming that $\omega_0 > \gamma/2$. Use it to obtain a solution to the equation

$$\ddot{\eta} + \gamma \dot{\eta} + \omega_0^2 \eta = A e^{-\alpha t} \Theta(t),$$

where $\Theta(t)$ is the Heaviside theta function, and show that the result corresponds to the choice of initial conditions $\eta(0) = 0$ and $\dot{\eta}(0) = 0$ in the expected general solution

$$\eta(t) = C_1 \cos(\omega t)e^{-\gamma t/2} + C_2 \sin(\omega t)e^{-\gamma t/2} + \frac{Ae^{-\alpha t}}{\alpha^2 - \gamma \alpha + \omega_0^2},$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$. [12]

(TURN OVER
Consider two charged particles of mass $m_1$ and $m_2$, charge $e_1$ and $e_2$ with $e_1 = -e_2 = e$, and position vectors $r_1$ and $r_2$ that are constrained to move in the $x - y$ plane in the presence of a magnetic field perpendicular to the plane, $B = B\hat{z}$. The two particles interact via the Coulomb potential $V(r) = -e^2/r$, $r = |r_1 - r_2|$.

(a) Introduce the centre of mass and relative position coordinates

$$R = \frac{m_1 r_1 + m_2 r_2}{M}, \quad r = r_2 - r_1, \quad M = m_1 + m_2,$$

and write the Lagrangian of the system in the gauge $A(r) = (B \times r)/2$. [Hint: it may be convenient to keep the electromagnetic potential $A$ in its implicit vectorial form $A = (B \times r)/2$ rather than explicitly writing out each component.]

Show that, up to a total time derivative that can be neglected, the Lagrangian can be written as

$$L = \frac{M}{2} \ddot{R}^2 + \frac{\mu}{2} \dot{r}^2 + \frac{e}{r} - \frac{e m_1 - m_2}{2M} \dot{r} \cdot (B \times r) - e\dot{R} \cdot (B \times r),$$

where $\mu = m_1 m_2 / M$ is the reduced mass. \[10\]

(b) Obtain the Hamiltonian of the system and show that it can be written as

$$H = \left[\frac{P + e(B \times r)}{2M}\right]^2 + \left[\frac{p + e^*(B \times r)}{2\mu}\right]^2 - \frac{e^2}{r},$$

where $e^* = e(m_1 - m_2)/2M$. Use the form of the Hamiltonian to show that the energy of the system and the momentum of the centre of mass are constants of the motion. \[6\]

(c) Working in the reference frame where $P = 0$, derive Hamilton’s equations of motion. [Note that since $r$ lies in the $x - y$ plane and $B$ is perpendicular to it, then $|B \times r| = Br$.] \[6\]

(d) Use the first order differential equations of motion to derive second order equations for $x$ and $y$ alone, $r = (x, y)$:

$$\begin{cases} 
\mu \ddot{x} + 2e^* B \dot{y} = -\frac{e^2 B^2}{M} x - \frac{e^2}{r^2} x \\
\mu \ddot{y} - 2e^* B \dot{x} = -\frac{e^2 B^2}{M} y - \frac{e^2}{r^2} y.
\end{cases}$$

Show that these equations admit a solution of the form

$$\begin{cases} 
x = R \cos(\omega t) \\
y = R \sin(\omega t),
\end{cases}$$

with $R$, $\omega$ constants. Comment on the corresponding motion of the two particles: is it consistent with what you would expect for two particles moving in a magnetic field and interacting via a centrosymmetric potential? Compute the dependence of $\omega$ on $B$ and $R$. \[11\]
A dynamical system with Hamiltonian $H$ is described by independent coordinates $q_i$ ($i = 1, ..., n$) and corresponding generalised (canonical) momenta $p_i$.

(a) Explain what is meant by the Poisson Bracket $\{ f, g \}$ of two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ that depend on the generalised coordinates $q_i$ and $p_i$ and on time $t$. [3]

Show that if one of the functions coincides with a coordinate $q_j$ or a momentum $p_j$, then the Poisson Bracket reduces to a partial derivative, and therefore that $\{q_i, p_j\} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta symbol. [3]

Starting from Hamilton’s equations of motion, show that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}. \quad [3]$$

Use the Jacobi Identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

to show that if $f$ and $g$ satisfy the relationships

$$\frac{\partial f}{\partial t} + \{f, H\} = 0, \quad \frac{\partial g}{\partial t} + \{g, H\} = 0,$$

then so does $h$ defined as $h = \{f, g\}$. [5]

(b) A new set of coordinates and momenta $(Q_i, P_i)$ is defined by

$$Q_i = Q_i(q_j, p_j), \quad P_i = P_i(q_j, p_j), \quad i = 1, ..., n.$$

What condition must the new coordinates satisfy in order that this transformation is canonical, i.e. preserves the form of Hamilton’s equations of motion? [3]

For a system with two degrees of freedom, two new coordinates are defined by

$$Q_1 = q_1^2, \quad Q_2 = q_1 + q_2.$$

Find the most general expressions for the new generalised momenta $P_1(q_1, q_2, p_1, p_2)$ and $P_2(q_1, q_2, p_1, p_2)$ such that the transformation is canonical. [12]

Find a particular choice for the $P_i$ that reduces the Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

to

$$H = P_1^2 + P_2. \quad [4]$$

(TURN OVER)
The Lagrangian density for a triplet of real scalar fields in 3 + 1 space-time dimensions, $\varphi_a(t, x_1, x_2, x_3)$ with $a = 1, 2, 3$, is

$$L = \frac{1}{2}(\partial_\mu \varphi_a)(\partial^\mu \varphi_a) - \frac{1}{2}\lambda \varphi_a \varphi_a,$$

where $\partial^\mu = (\partial/\partial t, -\partial/\partial x_1, -\partial/\partial x_2, -\partial/\partial x_3)$. Use the Euler-Lagrange equations to derive the equations of motion for the fields $\varphi_a$.

Show that $L$ is invariant under the infinitesimal SO(3) rotation by an angle $\theta$

$$\varphi_a \rightarrow \varphi_a + \theta \epsilon_{abc} n_b \varphi_c,$$

where $n_a$ is an arbitrary unit vector and $\epsilon_{abc}$ is the three-dimensional Levi-Civita symbol, i.e. $\epsilon_{abc}$ is 1 if $(a, b, c)$ is an even permutation of $(1, 2, 3)$, $-1$ if it is an odd permutation, and 0 if any index is repeated.

Derive from first principles the Noether current $J^\mu$ corresponding to this symmetry of the Lagrangian density.

Deduce that the three quantities

$$Q_a = \int d^3x \epsilon_{abc} \frac{\partial}{\partial t} \varphi_b \varphi_c,$$

are all conserved and verify this directly using the field equations satisfied by the $\varphi_a$. You should state explicitly any assumptions needed for this result to hold.

The non-linear version of the Klein-Gordon Lagrangian density for a scalar field $\phi(x, t)$ is given by

$$L = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + F(\phi),$$

where $F(\phi)$ is a differentiable function of its argument.

(a) Show that the Euler-Lagrange equation for the system leads to the equation of motion

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + f(\phi),$$

where $f(\phi) = F'(\phi)$.

If $\phi = \phi(x, t)$ is a solution of this equation, show that the function

$$\phi_1 = \phi(x \cosh \beta + t \sinh \beta, t \cosh \beta + x \sinh \beta),$$

where $\beta$ is an arbitrary constant, is also a solution.

(b) Consider the particular case $f(\phi) = -a \phi + b \phi^n$, for positive constants $a, b$ and integer $n > 1$. Determine the values of constants $A$ and $B$ for which the function

$$w(x) = \left[ A \cosh^2 (Bx) \right]^{1/n}$$

(TURN OVER for continuation of question 5
is a (static) solution of the equation of motion. Sketch this solution for $-\infty < x < +\infty$ and several different values of $n$. Hence show that

$$\phi(x, t) = w(x \cosh \beta - t \sinh \beta),$$

where $\beta$ is a positive constant, is a travelling-wave solution and describe its dependence on $x$ and $t$. [5]

6 The Landau free energy expansion for a uniaxial ferromagnet in a magnetic field can be written as

$$F = F_0 - hm + \frac{a}{2}m^2 + \frac{b}{4}m^4,$$

where $m$ is the magnetisation of the system and $h$ represents an externally applied magnetic field.

(a) Briefly discuss the origin of this expansion and what you know a priori about (some of) the terms and their coefficients. [4]

(b) Define and compute the exponent $\delta$ along the critical isotherm. [6]

(c) Compute the susceptibility $\chi = (\partial m/\partial h)_{h=0}$ as a function of $t = (T - T_c)/T_c$ both above ($t > 0$) and below ($t < 0$) the transition. Show that

$$\lim_{t \to 0^+} \frac{\chi(t)}{\chi(-t)} = 2.$$ [8]

(d) Add the term $d m^3/3$ to the free energy $F$ for a generic real parameter $d$ and set $h = 0$. Discuss how the nature of the ordering transition is affected (you may restrict the discussion to values of $d < 9ab/2$ as the solution for larger values of $d$ becomes more involved). [15]

END OF PAPER