THEORETICAL PHYSICS I

Answer three questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.
1. A simple pendulum of mass $m_2$ is free to oscillate in the vertical plane $x - y$. At its point of support the pendulum is attached to a mass $m_1$ which is free to move along the line $y = 0$.

![Diagram of a simple pendulum](image)

(a) Show that the Lagrangian for this system is

$$L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 \left( l^2 \dot{\phi}^2 + 2 l \dot{x} \dot{\phi} \cos \phi \right) + m_2 gl \cos \phi,$$

where $\phi$ is the angular displacement of the pendulum and $x$ is the horizontal position of the mass $m_1$, as shown in the figure. [8]

(b) Deduce the canonical momenta $p_x$ and $p_\phi$ conjugate to the generalised coordinates $x$ and $\phi$ and show that $p_x$ is a conserved quantity. [6]

(c) Show that the path of $m_2$ is the arc of an ellipse if $p_x = 0$. [10]

(d) For the case considered in (c) derive an expression for the energy $E$ of the system and use it to show that the time $t$ taken for the pendulum to move from angle $\phi_1$ to $\phi_2$ within a single oscillation is given by

$$t = l \sqrt{\frac{m_2}{2(m_2 + m_1)}} \int_{\phi_1}^{\phi_2} d\phi \sqrt{\frac{m_1 + m_2 \sin^2 \phi}{E + m_2 gl \cos \phi}}.$$ [9]

2. A harmonic oscillator is weakly perturbed by a cubic potential $\lambda x^3$ so that its Hamiltonian has the form

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \lambda x^3,$$

where $\lambda$ is small.

(a) Find constraints on the parameters $\alpha_i, \beta_i$ which make the coordinate transformation

$$x = X + \alpha_1 X^2 + 2 \alpha_2 XP + \alpha_3 P^2$$
$$p = P + \beta_1 X^2 + 2 \beta_2 XP + \beta_3 P^2$$

canonical to first order in $\alpha_i$ and $\beta_i$. [8]
(b) Carry out the canonical transformation from part (a) on the Hamiltonian $H$ and find values for the parameters $\alpha_i, \beta_i$ in terms of $m, \omega$ and $\lambda$ which make the transformed Hamiltonian $K(X, P)$ harmonic to first order in $\alpha_i$ and $\beta_i$, i.e.

$$K(X, P) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 + O(\alpha_i^2, \beta_i^2),$$

and state the resulting canonical transformations. [10]

(c) Use Hamilton’s equations for $K$ to find expressions for $X(t)$ and $P(t)$ to first order in $\alpha_i$ and $\beta_i$. [6]

(d) Use your answers to parts (b) and (c) to find expressions for $x(t)$ and $p(t)$ and comment on the effect of the perturbation. [9]

3 Show explicitly that the Lagrangian

$$L = \frac{1}{2}mv^2 + ev \cdot A - e\phi$$

yields the correct equation of motion for a particle of (positive) charge $e$ and mass $m$ moving in an electromagnetic field:

$$E = -\nabla \phi - \frac{\partial A}{\partial t} \quad \text{and} \quad B = \nabla \times A,$$

where $A$ and $\phi$ are the usual electromagnetic potential functions. [7]

Explain what is meant by gauge invariance in this context. [3]

In terms of cylindrical coordinates $(r, \theta, z)$, the potential functions are $\phi = \lambda z^2$ and $A = (0, \mu r, 0)$, where $\lambda$ and $\mu$ are positive constants.

(a) Use the Euler-Lagrange equations to derive the (three) equations of motion of the particle. [7]

(b) Determine the total energy of the particle and show that it is a constant of the motion. [5]

(c) Show that the Euler-Lagrange equation for $\theta(t)$ gives rise to a second constant of the motion. [3]

(d) Describe the motion of the particle given that $r$ is constant, $r = R$, and the angular velocity is non-zero, $\dot{\theta} \neq 0$. [4]

(e) Explain the significance of the special case $\lambda = (2e\mu^2/m)n^2$, where $n$ is an integer. [4]
The Klein-Gordon Lagrangian density for a real scalar field $\phi(x, t)$ is

$$L_{KG}[\phi] = \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2,$$

where $\partial^\mu$ represents the differential operator ($\partial/\partial t, -\nabla$). Use the Euler-Lagrange equations to derive the equation of motion:

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0.$$

The Fourier transformed field $\tilde{\phi}(k, t)$ is defined by

$$\phi(x, t) = \int d^3k \tilde{\phi}(k, t) e^{ik \cdot x}.$$

Find and solve the equation of motion satisfied by $\tilde{\phi}(k, t)$.

A dynamical system is described by two real scalar fields, $\phi_1$ and $\phi_2$, with Lagrangian density

$$L = L_{KG}[\phi_1] + L_{KG}[\phi_2] + L_{int},$$

where the interaction term is $L_{int} = g \phi_1 \phi_2$, with $g$ a real constant, $0 < g < m^2$.

Derive the (two) coupled equations of motion for the system.

Solve these equations to obtain general solutions in terms of the Fourier transformed fields $\tilde{\phi}_i(k, t)$.

At time $t = 0$, the system is in a state corresponding to

$$\phi_1 = A \sin(q \cdot x), \quad \frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_2}{\partial t} = \phi_2 = 0,$$

with $A$ and $q$ a constant scalar and vector respectively. Find $\phi_1$ and $\phi_2$ for $t > 0$.

State Noether’s theorem and explain its significance.

A Lagrangian density $L$ is a functional of a scalar field $\phi(x, t)$. If the Lagrangian is invariant under an infinitesimal field transformation of the form

$$\phi \rightarrow \tilde{\phi} = \phi + \delta \phi,$$

show that there is a continuity equation

$$\frac{\partial J_x}{\partial x} + \frac{\partial \rho}{\partial t} = 0,$$

where

$$\rho = \frac{\partial L}{\partial \tilde{\phi}'} \delta \phi, \quad J_x = \frac{\partial L}{\partial \phi'} \delta \phi,$$

and $\phi$ and $\phi'$ denote partial differentiation with respect to $t$ and $x$ respectively.

Generalising to 3 spatial dimensions, and using covariant notation, show that this corresponds to conservation of the Noether current $J^\mu$, i.e. $\partial_\mu J^\mu = 0$, where

$$J^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi.$$
The Lagrangian density for a scalar field in $n$ space-time dimensions, \( \varphi(t, x_1, x_2, ..., x_{n-1}) \), is

\[
\mathcal{L} = \frac{1}{2} (\partial_{\mu} \varphi)(\partial^\mu \varphi) - \lambda \varphi^4,
\]

where \( \partial^\mu = (\partial/\partial t, -\partial/\partial x_1, -\partial/\partial x_2, ..., -\partial/\partial x_{n-1}) \) and hence \( \partial_{\mu} x^\mu = n \). Use the Euler-Lagrange equations to derive the equation of motion

\[
\partial^\mu \partial_{\mu} \varphi + 4 \lambda \varphi^3 = 0.
\]

A current \( J^\mu \) is defined by

\[
J^\mu = (\varphi + x^\nu \partial_\nu \varphi) \partial_\mu \varphi - \partial_\mu \mathcal{L}.
\]

Show that

\[
\partial_\mu J^\mu = (n - 4) \mathcal{L},
\]

and hence that \( J^\mu \) is a conserved current only in 4 space-time dimensions.

An infinite one-dimensional system has a temperature distribution \( T(x, t) \) given by the heat transmission equation

\[
-\frac{\partial^2 T}{\partial x^2} + 2\alpha \frac{\partial T}{\partial t} + \frac{1}{c^2} \frac{\partial^2 T}{\partial t^2} = s(x, t),
\]

where \( s(x, t) \) is a heat source, and \( \alpha \) and \( c \) are positive constants.

(a) Use Fourier methods to show that the Green’s function

\[
G(k, t - t') = \int_{-\infty}^{\infty} e^{-ik(x-x')} G(x, x'; t, t') \, dx
\]

for this heat equation has the form

\[
G(k, t - t') = \begin{cases} 
0 & t < t' \\
\frac{1}{\sqrt{\alpha^2 - k^2/c^2}} e^{-\alpha c^2(t-t')} \sinh \sqrt{\alpha^2 c^2 - k^2 c^2}(t-t') & t > t'
\end{cases}
\]

and comment on the result.

(b) Find the temperature \( T(x, t) \) of the system if \( s(x, t) = \cos(px) \delta(t-t_0) \) and \( T(x, t < t_0) = 0 \), for the two cases \( \alpha > p/c \) and \( \alpha < p/c \) and discuss your results.